Waves 1

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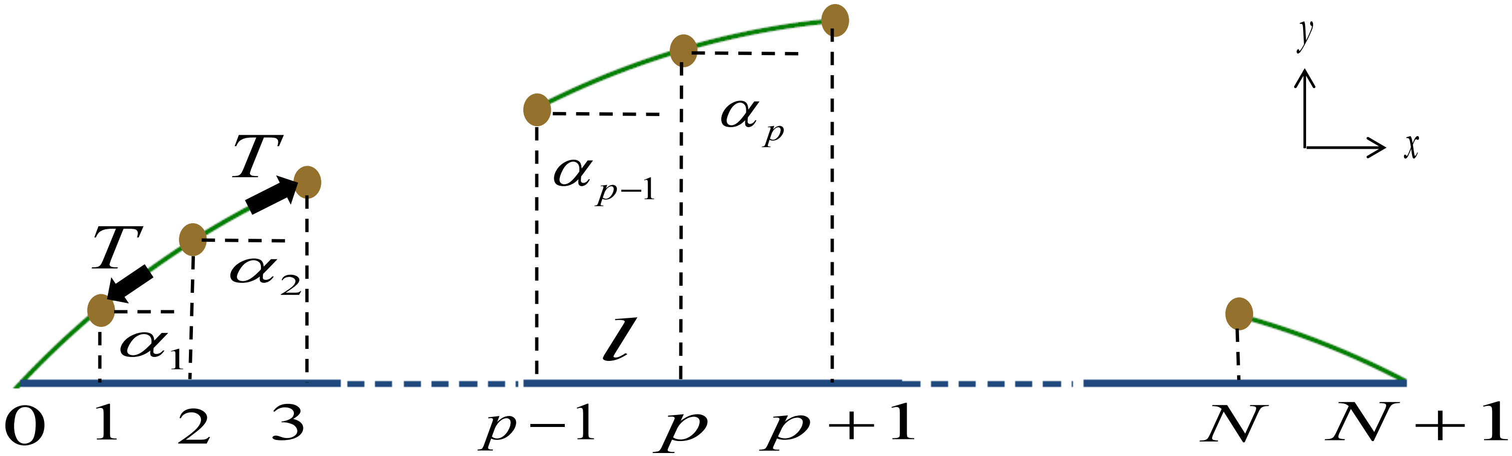
1. N coupled oscillators – towards the continuous limit
2. Stretched string and the wave equation
3. The d'Alembert solution
4. Sinusoidal waves, wave characteristics and notation

N coupled oscillators

Consider flexible elastic string to which are attached *N* particles of mass *m*, each a distance *l* apart. The string is fixed at each end. Small transverse displacements are applied → transverse oscillations



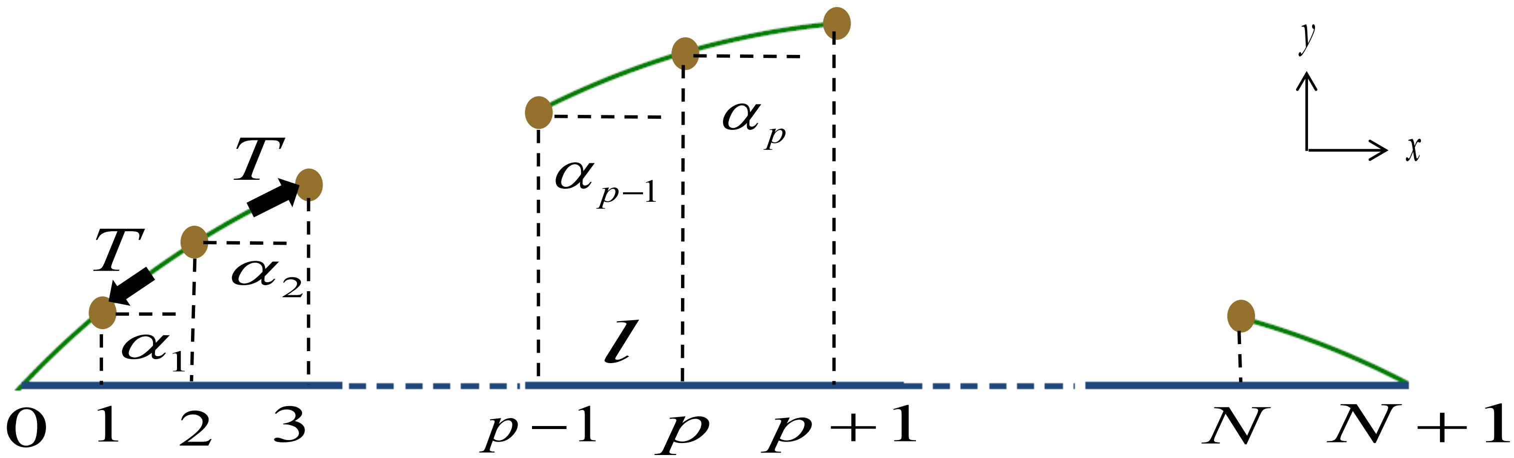
N coupled oscillators: special cases

Let's first consider the special cases *N=1* and *N=2*

N coupled oscillators: general case

Now let’s try and find solution for a general value *N*

N coupled oscillators: the solution



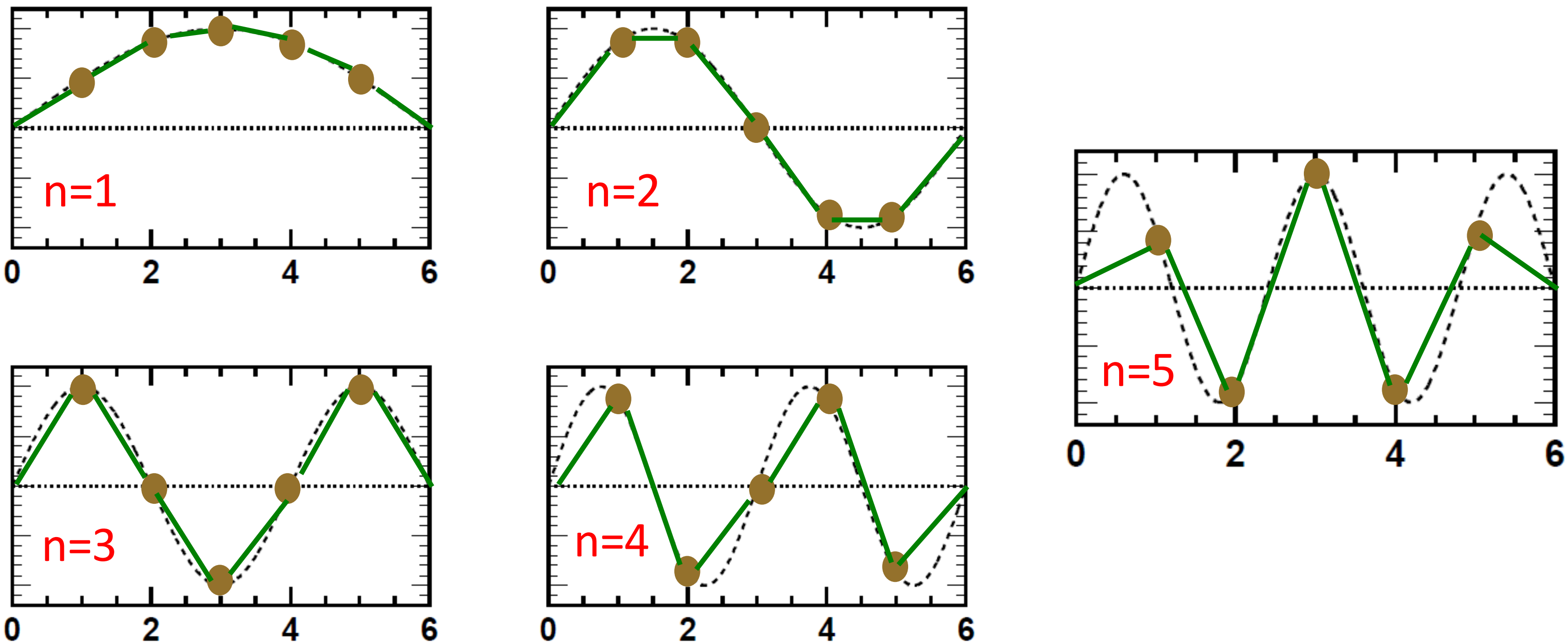
Displacement for mass *p* when oscillating in mode *n* and angular frequency:

$$\boxed{y_{pn}(t) = C_n \sin\left(\frac{pn\pi}{N+1}\right)\cos(\omega_n t + \phi_n)} \qquad \boxed{\omega_n = 2\omega_0 \sin\left[\frac{n\pi}{2(N+1)}\right]} \qquad \omega_0 = \sqrt{T/ml}$$

Although the value of *n* can go beyond *N*, this just generates duplicate solutions, *i.e.* there are *N* normal modes in total.

N coupled oscillators: modes for N=5

Look at each mode for *N=5*, with snapshot taken at *t=0*

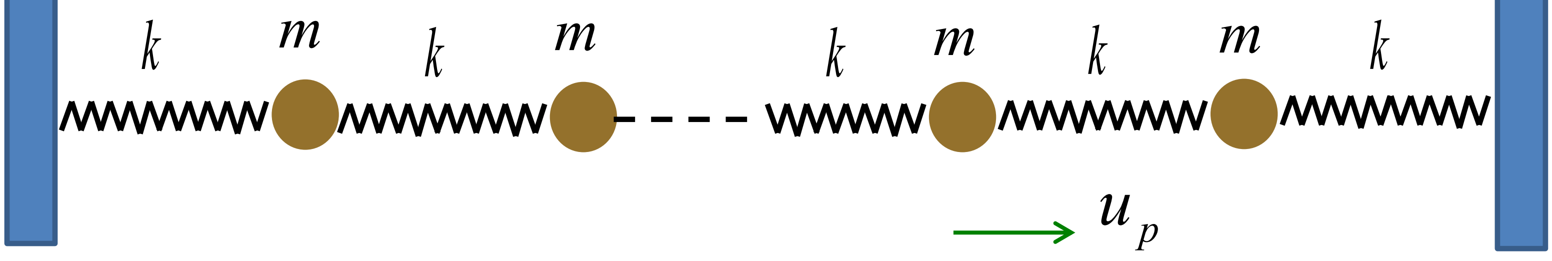


Note how the displacement of every particle falls on a sine curve!

N coupled oscillators: N very large

Let's explore the scenario where *N* is very large, which starts to approximate case of a real, continuous, string

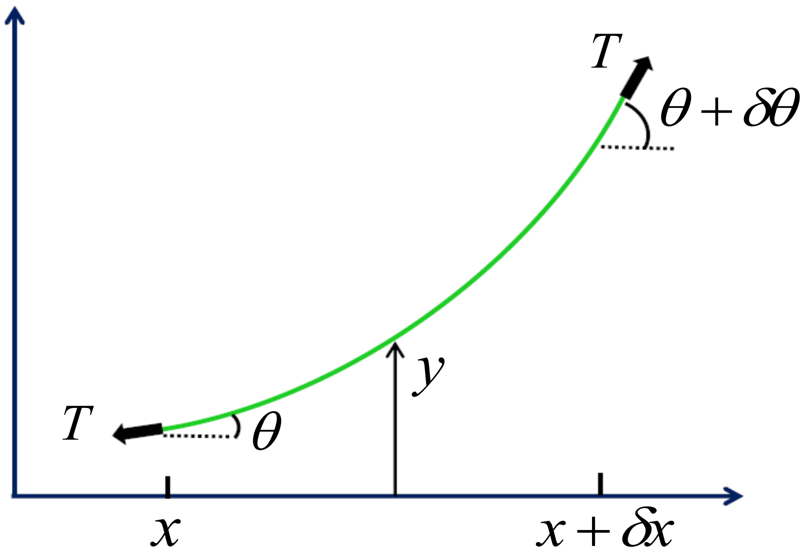
System of springs and N masses: longitudinal oscillations



Let u_p be displacement from equilibrium position of mass p

Stretched string

Consider a segment of string of linear density ρ stretched under tension T



$\theta, \delta\theta$

small

Stretched string and wave equation

Will show that the displacements on a stretched string obey

$$\boxed{\frac{\partial^2 y}{\partial x^2} = \left(\frac{\rho}{T}\right)\frac{\partial^2 y}{\partial t^2}}$$

which is the wave equation

$$\boxed{\frac{\partial^2 y}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 y}{\partial t^2}} \quad \text{with} \quad \boxed{c = \sqrt{\frac{T}{\rho}}}$$

Jean-Baptiste le Rond d'Alembert

- 1717-1783
- Lived in Paris
- Mathematician and physicist
- Also a music theorist and co-editor with Diderot of a famous encyclopaedia

d'Alembert solution of wave equation

We will show how the wave equation can be solved to yield solutions of form:

$$\boxed{y(x,t) = f(x-ct) + g(x+ct)}$$

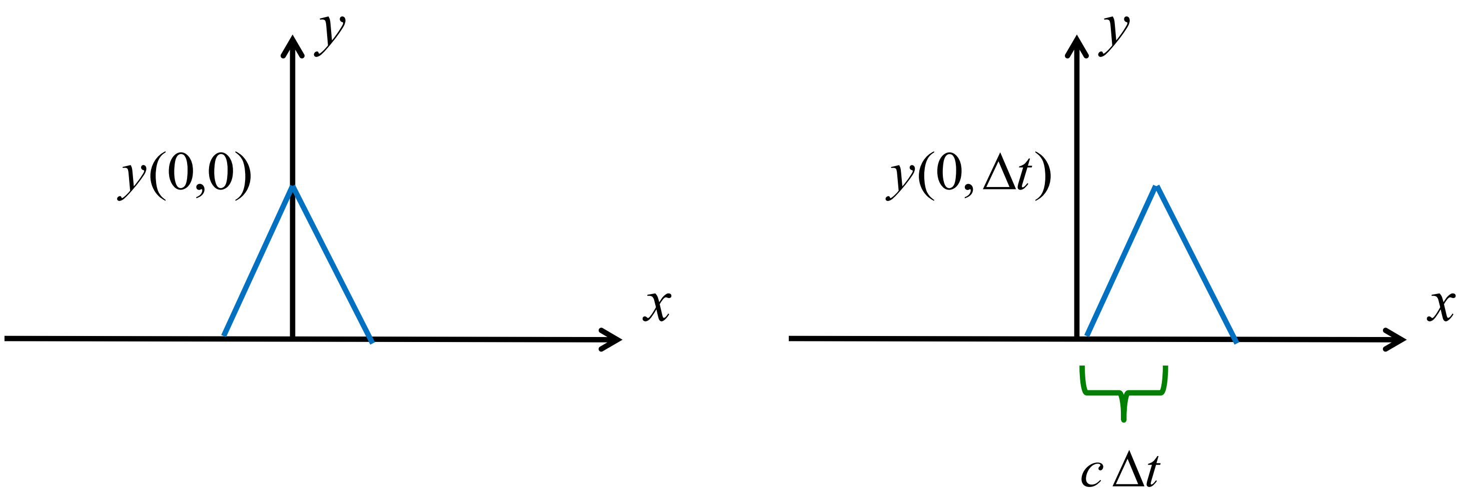
Here *f* and *g* are any functions of *(x-ct)* & *(x+ct)*, determined by initial conditions.

We will then interpret this solution.

Interpretation of D'Alembert solution

$$y(x,t) = f(x - ct)$$

Focus on x=0 and consider situations at *t=0* and *t=Δt*

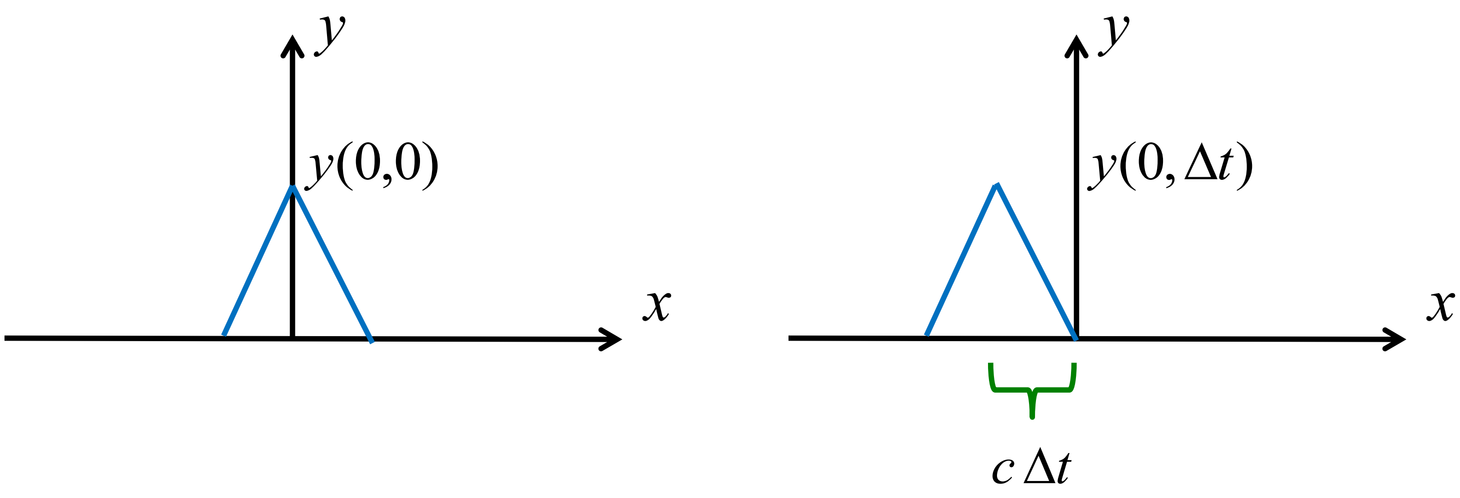


Wave moves to right with speed c

Interpretation of D'Alembert solution

$$y(x,t) = g(x+ct)$$

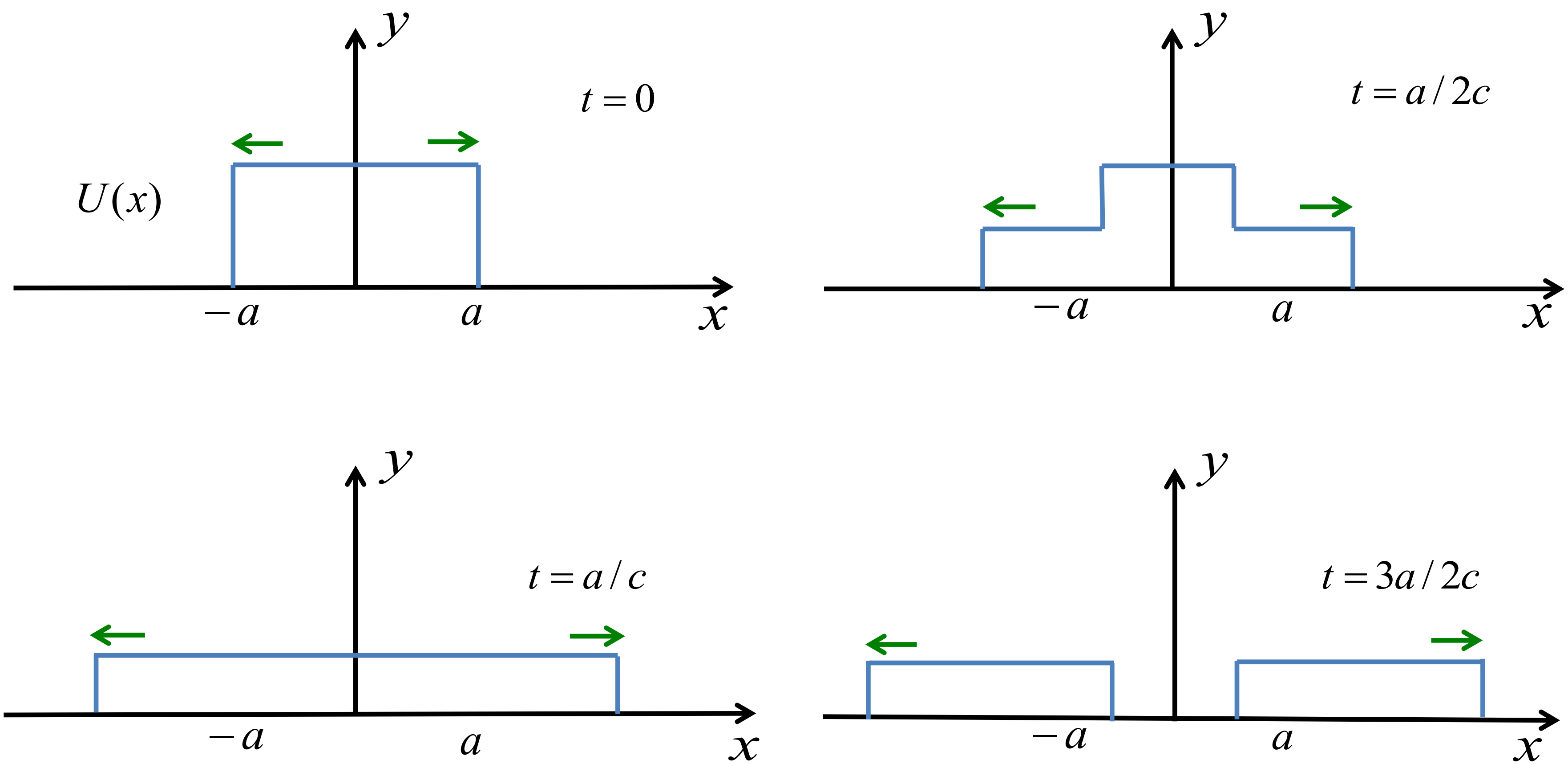
Focus on x=0 and consider situations at *t=0* and *t=Δt*



Wave moves to left with speed c

d'Alembert's solution with boundary conditions

Example: rectangular wave of length 2a released from rest $\Rightarrow y(x,t) = \frac{1}{2}[U(x-ct)+U(x+ct)]$



Sinusoidal waves

A very common functional dependence for *f* and *g*...

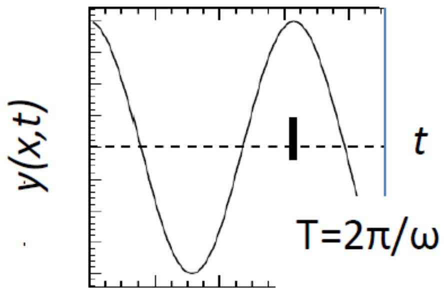
$$y(x,t) = f(x - ct) + g(x + ct)$$

...is sinusoidal. In this case it is usual to write:

$$\boxed{y(x,t) = A\cos(kx - \omega t) + B\cos(kx + \omega t)}$$

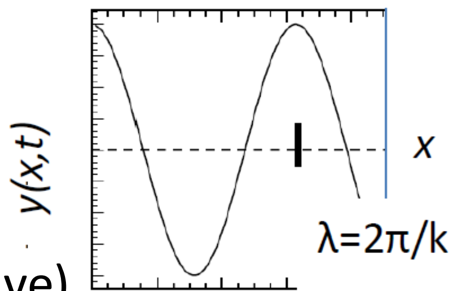
with *k* and *ω* (and *A* and *B*) constants

or *A*sin*(kx-ωt)* ... etc (choice doesn't matter, unless we are comparing one wave with another and then relative phases become important)



- **speed** of wave $\boxed{c = \omega / k}$
- **frequency** $\boxed{f = 1/T = \omega / 2\pi}$

  where ω is **angular frequency**



- **wavelength** $\boxed{\lambda = 2\pi / k}$

  where k is the **wave-number**
  (or **wave-vector** if also used to indicate direction of wave)

Notation choices

Sinusoidal solution $y(x,t) = A\cos(kx - \omega t)$ (writing here, for compactness, only the forward-going solution)

Using the relationships between *k,ω, λ* & *c* this can be expressed in many forms

$$y(x,t) = A\cos[k(x - ct)]$$

Also note that sometimes it is convenient to write $y(x,t) = A\cos(\omega t - kx)$
Changes nothing (for cosine, trivially so, & practically not even for sine function, as overall sign can be absorbed in constant) & still describes forward-going wave.

A very frequent approach is to use complex notation (we already made use of this when analysing normal modes, and you will have seen it in circuit analysis)

$$y(x,t) = \mathrm{Re}\big[A\exp[i(kx - \omega t)]\big]$$

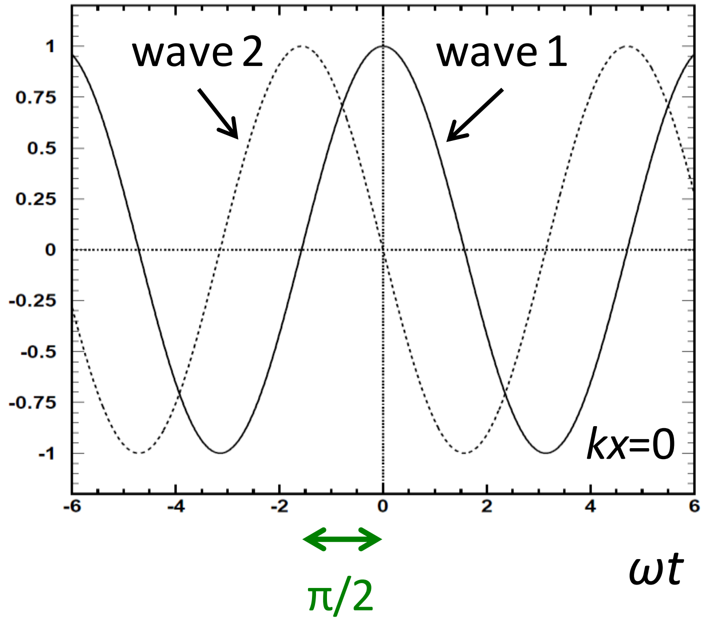
or $y(x,t) = \mathrm{Im}\big[A\exp[i(kx - \omega t)]\big]$ if its important to pick out sine function.
Note that often the 'Re' or 'Im' is implicit, and it gets omitted in discussion.

Phase differences

Often important to specify phase shifts. Only meaningful to do so when we are comparing one wave to another.

$$y_1(x,t) = A\cos(kx - \omega t)$$

$$y_2(x,t) = A\cos(kx - \omega t + \phi)$$



In this example wave 2 leads wave 1 by π/2, i.e. ϕ=-π/2

Can be expressed with complex notation

$$y_2(x,t) = \mathrm{Re}\left[A\exp[i(kx - \omega t + \phi)]\right]$$

Nicer still to subsume phase into amplitude

$$y_2(x,t) = \mathrm{Re}\left[A\exp[i(kx - \omega t)]\right] \quad \text{with} \quad A = |A|\exp(i\phi)$$