Wave Motion: revision

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Trinity Term 2014

Introduction

Two lectures to remind ourselves of what we learned last term

Will restrict discussion to the topics on the syllabus

Will re-state main relations/facts/principles you should know.

Most of these I will not re-derive (no time) – look back to last term's notes.

Essential information contained in these slides, but some points will be augmented by working through on board.

Will illustrate with some typical problems from recent prelims papers. (No guarantee my answers are correct! If working through them we find bugs I shall fix the online version of the slides)

These slides & HT material on http://www.physics.ox.ac.uk/users/wilkinsong

Normal modes

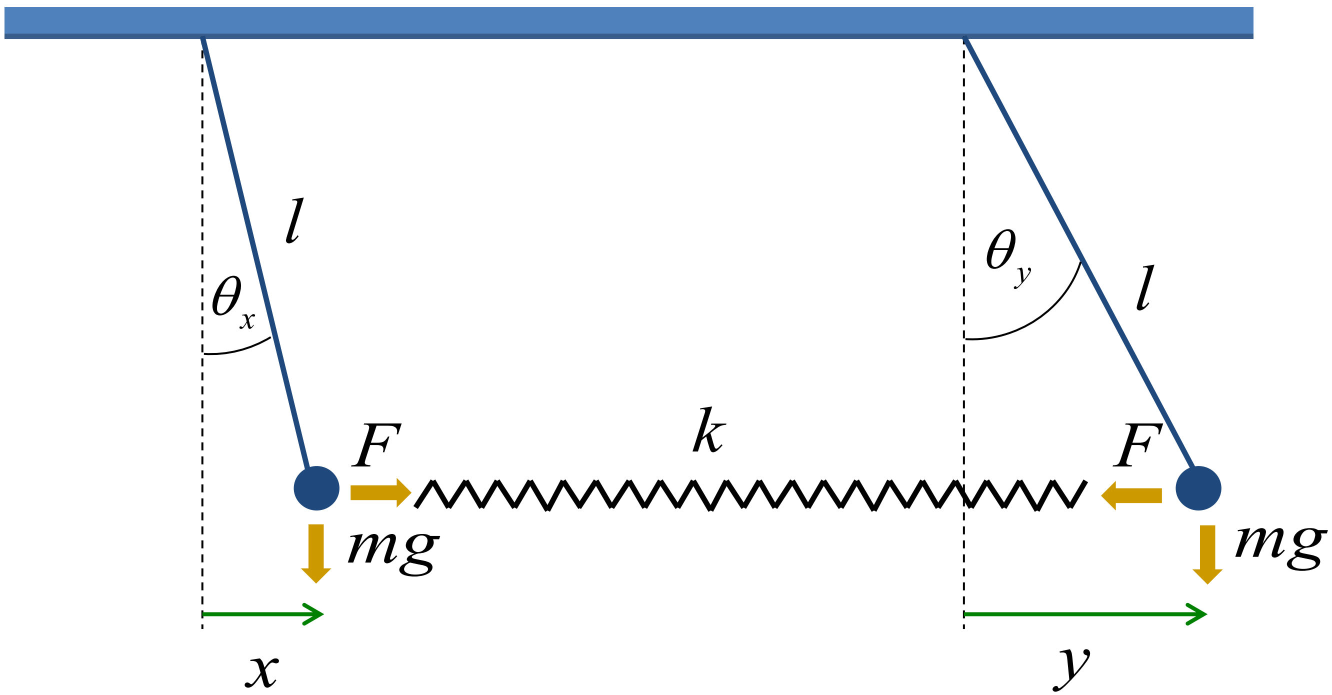
Normal modes - syllabus

According to the course handbook you should know about the following

Coupled undamped oscillations in systems with two degrees of freedom. Normal frequencies, and amplitude ratios in normal modes. General solution (for two coupled oscillators) as a superposition of modes. Total energy, and individual mode energies. Response to a sinusoidal driving term.

Let's remind ourselves of the essentials, before looking at a few past problems

An illustrative example: coupled pendula



First thing to do is to write down the equations of motion:

$$m\ddot{x} = -mg\frac{x}{l} + k(y-x)$$

$$m\ddot{y} = -mg\frac{y}{l} - k(y-x)$$

Finding the normal modes

Recall there are two good ways to solve these problems:

1. The decoupling method

The equations are decoupled by finding the 'mode (or normal) coordinates'. If the system is symmetric (*i.e.* equal masses, spring constants, length of pendula...) then a good bet will be

$$q_1 \equiv \frac{1}{\sqrt{2}}(x+y)$$

$$q_2 \equiv \frac{1}{\sqrt{2}}(x-y)$$

(the factor of 1/√2 is not necessary for solving the problem, and is often omitted. It's there to make other expressions, *i.e*. those for the energy, come out right)

In practice then one adds and subtracts the equations of motion.

If the system is not symmetric then the normal modes will in general be more complicated. In that case I advise a second plan of attack...

2. The matrix method

I'll remind you of this shortly.

Solving with decoupling method

So for the coupled pendula we add and subtract the equations of motion:

$$m\ddot{x} = -mg\frac{x}{l} + k(y-x) \qquad m\ddot{y} = -mg\frac{y}{l} - k(y-x)$$

to obtain

$$\ddot{q}_1 = -\omega_1^2 q_1 \quad \text{with } \omega_1^2 = \frac{g}{l} \qquad \ddot{q}_2 = -\omega_2^2 q_2 \quad \text{with } \omega_2^2 = \frac{g}{l} + 2\frac{k}{m}$$

$$q_1 \equiv x + y \qquad q_2 \equiv x - y$$

the solutions of which are clearly

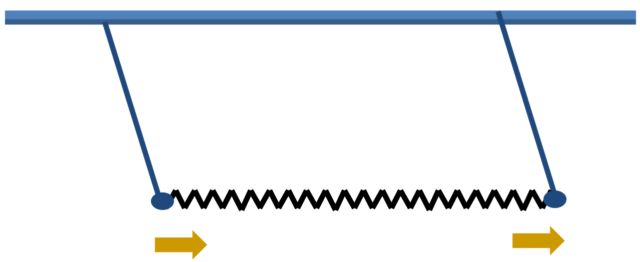
$$\boxed{q_1 = A_1\cos(\omega_1 t + \varphi_1)} \qquad \boxed{q_2 = A_2\cos(\omega_2 t + \varphi_2)}$$

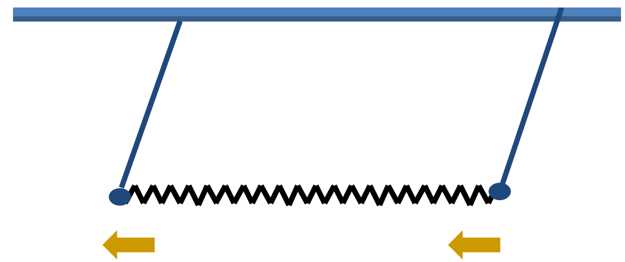
where $A_{1,2}$ & $\phi_{1,2}$ are constants set by inital conditions

Coupled pendula – the normal modes

First normal mode: centre-of-mass motion

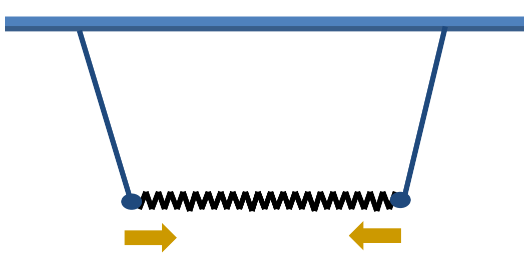
$$q_1 = A_1 \cos(\omega_1 t + \varphi_1) \qquad q_1 \equiv x + y \qquad \omega_1^2 = \frac{g}{l}$$

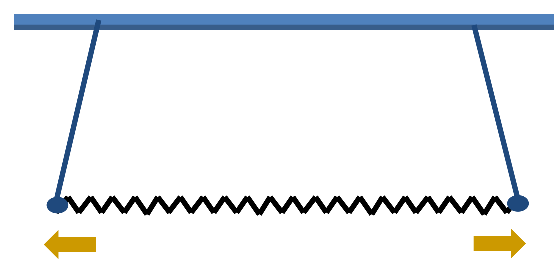




Second normal mode: relative motion

$$q_2 = A_2 \cos(\omega_2 t + \varphi_2) \qquad q_2 \equiv x - y \qquad \omega_2^2 = \frac{g}{l} + 2\frac{k}{m}$$





This correlated and anti-correlated motion is given by the form of the normal coordinates and is typical for the simple problems we will encounter.

Coupled pendula: the general solution

General solution is a sum of the two normal modes

$$x = A_1 \cos(\omega_1 t + \phi_1) + A_2 \cos(\omega_2 t + \phi_2)$$
$$y = A_1 \cos(\omega_1 t + \phi_1) - A_2 \cos(\omega_2 t + \phi_2)$$

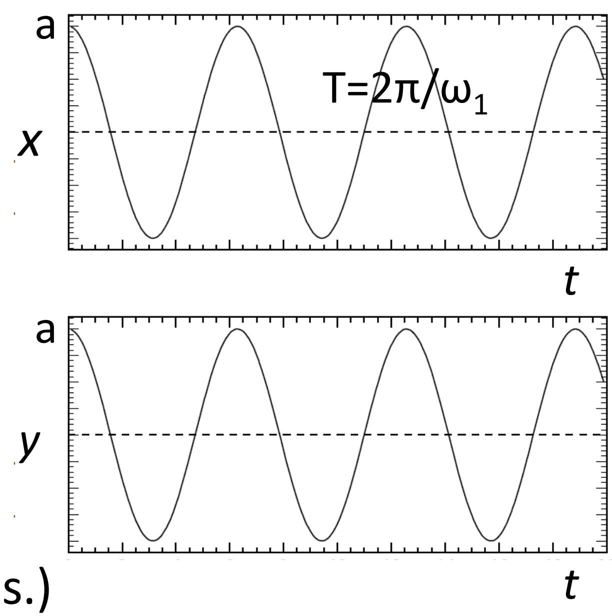
The constants A_1 ϕ_1, A_2 and ϕ_2 are set by the initial conditions

For example

$$x(0) = y(0) = a \quad ; \quad \dot{x}(0) = \dot{y}(0) = 0$$

gives

$$A_1 = a \quad ; \quad A_2 = 0 \quad ; \quad \phi_1 = 0$$



In this case only the 1st normal mode is excited.
(For other possibilities look back at HT lecture notes.)

Solving with matrix method

$$m\ddot{x} = -mg\frac{x}{l} + k(y-x)$$

$$m\ddot{y} = -mg\frac{y}{l} - k(y-x)$$

$\Rightarrow$

$$\begin{pmatrix} \frac{d^2}{dt^2} + \frac{g}{l} + \frac{k}{m} & -\frac{k}{m} \\ -\frac{k}{m} & \frac{d^2}{dt^2} + \frac{g}{l} + \frac{k}{m} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Expecting an oscillatory solution, so let's try substituting one in, making use of complex notation $\Rightarrow$

$$\begin{pmatrix} x \\ y \end{pmatrix} = \mathrm{Re}\begin{pmatrix} X \\ Y \end{pmatrix} e^{i\omega t}$$

X & Y are complex constants

We obtain:

$$\begin{pmatrix} -\omega^2 + \frac{g}{l} + \frac{k}{m} & -\frac{k}{m} \\ -\frac{k}{m} & -\omega^2 + \frac{g}{l} + \frac{k}{m} \end{pmatrix} \begin{pmatrix} X \\ Y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

eigenvector equation

Solving with matrix method

We have equation of sort $A\Psi = 0$

Non-trivial solution requires matrix has no inverse $\Rightarrow \det[A] = 0$

So here:

$$\begin{vmatrix} -\omega^2 + \frac{g}{l} + \frac{k}{m} & -\frac{k}{m} \\ -\frac{k}{m} & -\omega^2 + \frac{g}{l} + \frac{k}{m} \end{vmatrix} = 0$$

Eigenvalue equation

$$\left(-\omega^2 + \frac{g}{l} + \frac{k}{m}\right) = \pm\frac{k}{m} \longrightarrow \begin{array}{l} \omega_1^2 = \frac{g}{l} \\ \omega_2^2 = \frac{g}{l} + 2\frac{k}{m} \end{array}$$

Substitute these back into the eigenvector equ^n to find the *amplitude ratios*, *i.e.* the relationship between *X* and *Y*. Find *X*=*Y* and *X*=-*Y*, as before.

Illustrate these methods with a couple of past exam questions

We have reminded ourselves about:

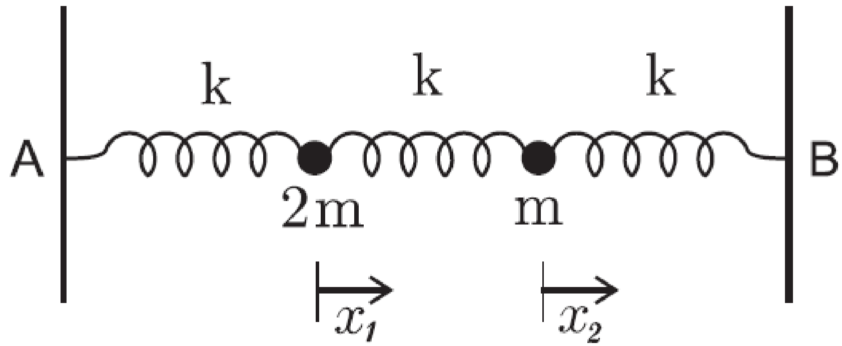
Coupled undamped oscillations in systems with two degrees of freedom. Normal frequencies, and amplitude ratios in normal modes. General solution (for two coupled oscillators) as a superposition of modes. Total energy, and individual mode energies. Response to a sinusoidal driving term.

Let's apply these techniques to

- TT 2008, Q11
- TT 2011, Q8

Trinity Term 2009, Q11

11. Two masses $2m$ and m are connected between two fixed points A and B with three identical massless springs, each of spring constant k. The masses $2m$ and m are free to execute small oscillations along the line of the springs, with displacements from their equilibrium positions of x_1 and x_2, respectively.



Show that the angular frequencies of the normal modes are given by

$$\omega_{\pm}^2 = \frac{k}{2m}(3 \pm \sqrt{3})\,.$$ [10]

Find the ratio of the displacements of the two masses for each normal mode. [4]

At time $t = 0$, the masses are at rest and have displacements $x_1 = x_0$ and $x_2 = 0$. Find the subsequent displacements as a function of time. [6]

Trinity Term 2009, Q11

Equations of motion:

$$2m\ddot{x}_1 = -k(2x_1 - x_2)$$
$$m\ddot{x}_2 = -k(2x_2 - x_1)$$

Different masses involved, so recommend matrix method.

Usual ansatz $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \mathrm{Re}\begin{pmatrix} X_1 \\ X_2 \end{pmatrix} e^{i\omega t}$ yields eigenvector equation

$$\begin{pmatrix} -\omega^2 + \dfrac{k}{m} & -\dfrac{k}{2m} \\ -\dfrac{k}{m} & -\omega^2 + \dfrac{2k}{m} \end{pmatrix}\begin{pmatrix} X_1 \\ X_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

which gives

$$\boxed{\omega_+^{\ 2} = \frac{k}{m}\left(\frac{3}{2} + \frac{\sqrt{3}}{2}\right) \qquad \left[\frac{X_1}{X_2}\right]_+ = -\frac{1}{1+\sqrt{3}}}$$

and

$$\boxed{\omega_-^{\ 2} = \frac{k}{m}\left(\frac{3}{2} - \frac{\sqrt{3}}{2}\right) \qquad \left[\frac{X_1}{X_2}\right]_- = \frac{1}{\sqrt{3}-1}}$$

with the X_1/X_2 amplitude ratios obtained by substituting ω^2 results back into eigenvector equation

Trinity Term 2009, Q11

So general solution is

$$x_1 = \frac{-A_+}{1+\sqrt{3}}\cos(\omega_+ t + \phi_+) + \frac{A_-}{\sqrt{3}-1}\cos(\omega_- t + \phi_-)$$

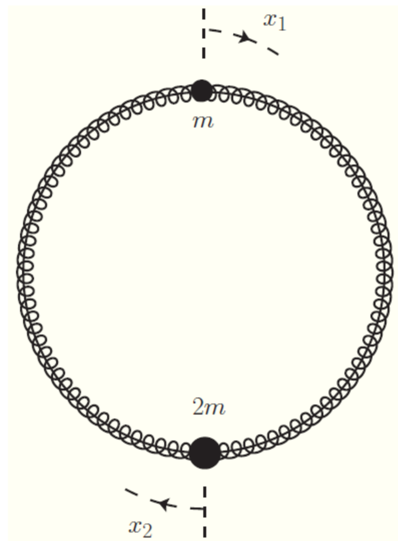
$$x_2 = A_+ \cos(\omega_+ t + \phi_+) + A_- \cos(\omega_- t + \phi_-)$$

Use initial conditions to fix $A_{+,-}$ and $\Phi_{+,-}$. It is the requirement that the masses start at rest that determines $\Phi_{+,-} = 0$, while the initial displacements fix $A_{+,-}$.

$$\Longrightarrow \quad \boxed{\begin{aligned} x_1 &= \frac{x_0}{\sqrt{3}}\left[\frac{1}{1+\sqrt{3}}\cos(\omega_+ t) + \frac{1}{\sqrt{3}-1}\cos(\omega_- t)\right] \\ x_2 &= \frac{x_0}{\sqrt{3}}\left[\cos(\omega_- t) - \cos(\omega_+ t)\right] \end{aligned}}$$

Trinity Term 2011, Q8

8. Two particles of mass m and $2m$ are free to slide on a frictionless horizontal circular wire of radius r. The particles are connected by two identical massless springs of spring constant k and natural length πr, which also wind around the wire, as shown in the figure.



(a) Write down the equations of motion for the displacements x_1 and x_2 of the two particles. [2]

(b) Find the frequencies of the normal modes of the system, and qualitatively explain their values. [6]

(c) Find the functional forms of the motions associated with each of the normal modes, including the appropriate time dependence. [7]

Trinity Term 2011, Q8

Equations of motion:
$$m\ddot{x}_1 = -2k(x_1 - x_2)$$
$$2m\ddot{x}_2 = -2k(x_2 - x_1)$$

Different masses involved, so recommend matrix method.

Usual ansatz $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \mathrm{Re}\begin{pmatrix} X_1 \\ X_2 \end{pmatrix} e^{i\omega t}$ yields eigenvector equation $\begin{pmatrix} -\omega^2 + \frac{2k}{m} & -\frac{2k}{m} \\ -\frac{k}{m} & -\omega^2 + \frac{k}{m} \end{pmatrix}\begin{pmatrix} X_1 \\ X_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

Solving this gives $\boxed{\omega = \sqrt{3k/m} \text{ in the case } X_1 = -2X_2}$

The other solution, with $\boxed{X_1 = X_2 \text{ is } \omega = 0}$. This is non-oscillatory !

So have
$$\boxed{\begin{aligned} x_1 &= A\cos\left[(\sqrt{3k/m})t + \Phi\right] \\ x_2 &= -\frac{A}{2}\cos\left[(\sqrt{3k/m})t + \Phi\right] \end{aligned}}$$
and
$$\boxed{\begin{aligned} x_1 &= B + vt \\ x_2 &= B + vt \end{aligned}}$$

with *A*,*B*,*Φ* and *v* constants to be determined from initial conditions

Trinity Term 2011, Q8

(d) If, at time $t = 0$, the displacements and velocities of the two particles are given by $x_1(0) = \pi r/10$, $x_2(0) = -\pi r/20$, and $\dot{x}_1(0) = \dot{x}_2(0) = \pi\omega r/20$ (where ω is the largest of the frequencies found in (b)), find the first time when the velocity of the particle of mass $2m$ is zero. Write your answer in terms of ω.

$$\boxed{\begin{aligned} x_1 &= \frac{\pi r}{10}\cos\left[\left(\sqrt{3k/m}\right)t\right] + \frac{\pi r}{20}\sqrt{\frac{3k}{m}}t \\ x_2 &= -\frac{\pi r}{20}\cos\left[\left(\sqrt{3k/m}\right)t\right] + \frac{\pi r}{20}\sqrt{\frac{3k}{m}}t \end{aligned}}$$

i.e. anticorrelated oscillations superimposed upon spinning motion

Velocity of mass 2m is zero when $\boxed{t = \sqrt{\frac{m}{3k}}\frac{3\pi}{2}}$

Other topics

Coupled undamped oscillations in systems with two degrees of freedom. Normal frequencies, and amplitude ratios in normal modes. General solution (for two coupled oscillators) as a superposition of modes. Total energy, and individual mode energies. Response to a sinusoidal driving term.

- Reminder about energy
- Driving term – TT2012, Q10

Energy of system with normal modes

Energy will include kinetic and potential contributions, with each coordinate featuring. So returning to the case of the two coupled pendula:

$$U = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2) \quad + \quad \frac{1}{2}m\left(\frac{g}{l} + \frac{k}{m}\right)(x^2 + y^2) - kxy$$

This a bit opaque. How does it look in terms of normal coordinates?

Recall (here with normalisation factors included):

$$x = \frac{1}{\sqrt{2}}(q_1 - q_2) \qquad \omega_1^2 = \frac{g}{l}$$

and

$$y = \frac{1}{\sqrt{2}}(q_1 + q_2) \qquad \omega_2^2 = \frac{g}{l} + 2\frac{k}{m}$$

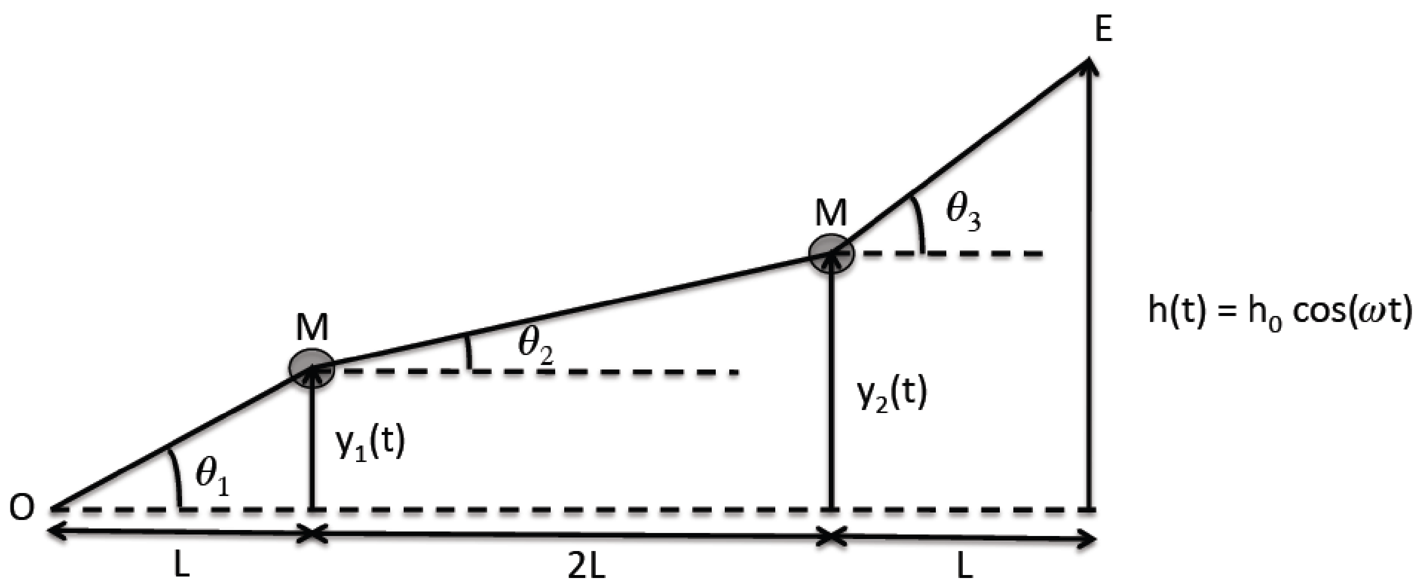
This gives a much more transparent expression, and without cross-terms:

$$\boxed{U = \underbrace{\left(\frac{1}{2}m\dot{q}_1{}^2 + \frac{1}{2}m\omega_1{}^2 q_1{}^2\right)}_{\text{energy in mode 1}} + \underbrace{\left(\frac{1}{2}m\dot{q}_2{}^2 + \frac{1}{2}m\omega_2{}^2 q_2{}^2\right)}_{\text{energy in mode 2}}}$$

So total energy of system = sum of energies in each excited mode

Trinity Term 2012, Q10

10. Two beads of identical mass M are placed at the same distance L from two ends of a string of total length $4L$ and negligible mass under uniform tension T. Damping is ignored. One extremity of the string, O, is attached to the wall, but the other extremity, E, is harmonically driven by a small transverse displacement $h(t) = h_0 \cos(\omega t)$ (see figure below).



(a) Calling the transverse displacements of the beads $y_1(t)$ and $y_2(t)$, derive the equation of motion for each mass, M, as a function of the angular frequency, $\omega_0 = \sqrt{T/(2ML)}$ and h_0. [7]

(b) Find the angular frequencies of the normal modes of oscillation associated with this system. [5]

Trinity Term 2012, Q10

Equations of motion: $$\boxed{\begin{aligned}\ddot{y}_1 &= -3\omega_0^2 y_1 \;+\; \omega_0^2 y_2 \\ \ddot{y}_2 &= y_1\omega_0^2 \;-\; 3\omega_0^2 y_2 + \;2\omega_0^2 h(t)\end{aligned}}$$ with $\omega_0^2 = T/2mL$

These are inhomogeneous, due to presence of *h(t)* term. To find the normal modes we must find the complementary functions, *i.e.* solve homogenous case:

$$\begin{aligned}\ddot{y}_1 &= -3\omega_0^2 y_1 \;+\; \omega_0^2 y_2 \\ \ddot{y}_2 &= y_1\omega_0^2 \;-\; 3\omega_0^2 y_2\end{aligned}$$

Symmetric system, so this is easily done by the decoupling method

$$\boxed{\begin{gathered}\ddot{p} = -2\omega_0^2 p \;\text{ with }\; p = y_1 + y_2 \\ \Rightarrow \omega_p = \sqrt{2}\omega_0\end{gathered}}$$ and $$\boxed{\begin{gathered}\ddot{q} = -4\omega_0^2 q \;\text{ with }\; q = y_1 - y_2 \\ \Rightarrow \omega_q = 2\omega_0\end{gathered}}$$

Trinity Term 2012, Q10

(c) Calculate the steady state amplitude of the displacement of each bead as a function of the driving frequency ω, h_0 and ω_0. What are the values of these amplitudes when $\omega = 0$? [4]

(d) Describe the motion of the system when the driving angular frequency $\omega = \sqrt{3}\omega_0$. [4]

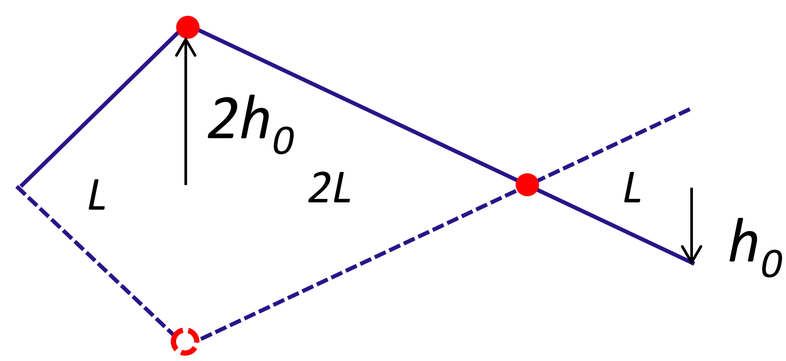
Finding steady state amplitude means finding the particular integral

$$\Rightarrow \boxed{\begin{aligned} y_1 &= \left[2\omega_0^4 h_0 / [(2\omega_0^2 - \omega^2)(4\omega_0^2 - \omega^2)]\right] \mathrm{Re}(e^{i\omega t}) \\ y_2 &= \left[2\omega_0^4 h_0 (3\omega_0^2 - \omega^2) / [(2\omega_0^2 - \omega^2)(4\omega_0^2 - \omega^2)]\right] \mathrm{Re}(e^{i\omega t}) \end{aligned}}$$

when $\omega = 0$ $\boxed{|y_1| = h_0/4 \text{ and } |y_2| = (3/4)h_0}$

when $\omega = \sqrt{3}\omega_0$

$$\boxed{y_1 = -2h_0 \mathrm{Re}(e^{i\omega t}) \text{ and } y_2 = 0}$$



Waves

Waves - syllabus

According to the course handbook you should know about the following

Derivation of the one-dimensional wave equation and its application to transverse waves on a stretched string. D'Alembert's solution. Sinusoidal solutions and their complex representation. Characteristics of wave motion in one dimension: amplitude, phase, frequency, wavelength, wavenumber, phase velocity. Energy in a vibrating string. Travelling waves: energy, power, impedance, reflection and transmission at a boundary. Superposition of two waves of different frequencies: beats and elementary discussion of construction of wave packets; qualitative discussion of dispersive media; group velocity. Method of separation of variables for the one-dimensional wave equation; separation constants. Modes of a string with fixed end points (standing waves): superposition of modes, energy as a sum of mode energies.

Let's remind ourselves of the essentials, before looking at a few past problems

Waves - syllabus

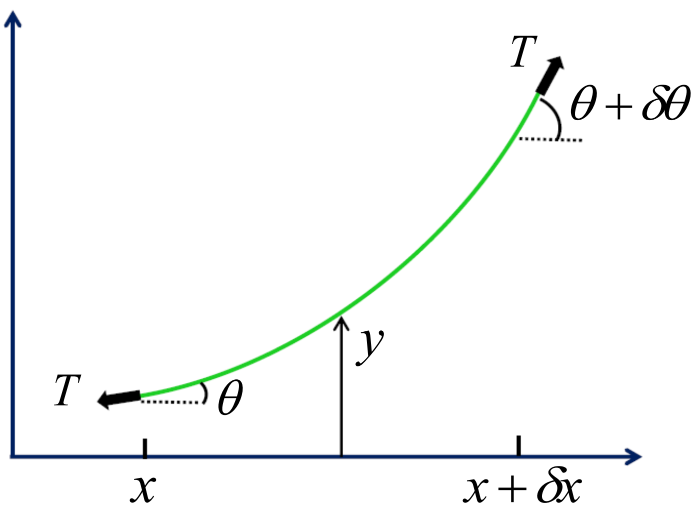
Focus first on the below topics

Derivation of the one-dimensional wave equation and its application to transverse waves on a stretched string. D'Alembert's solution. Sinusoidal solutions and their complex representation. Characteristics of wave motion in one dimension: amplitude, phase, frequency, wavelength, wavenumber, phase velocity. Energy in a vibrating string. Travelling waves: energy, power, impedance, reflection and transmission at a boundary. Superposition of two waves of different frequencies: beats and elementary discussion of construction of wave packets; qualitative discussion of dispersive media; group velocity. Method of separation of variables for the one-dimensional wave equation; separation constants. Modes of a string with fixed end points (standing waves): superposition of modes, energy as a sum of mode energies.

and illustrate by looking at Long Vacation 2011, Q9

Waves on a stretched string

Consider a segment of string of linear density *ρ* stretched under tension *T*



$\theta, \delta\theta$ small

Key point – means that no net horizontal force at first order, only vertical force, and we can make approximations such as $\sin\delta\vartheta \approx \delta\vartheta$

For full derivation see HT lecture notes or text book, *e.g.* French

$$\boxed{\frac{\partial^2 y}{\partial x^2} = \left(\frac{\rho}{T}\right)\frac{\partial^2 y}{\partial t^2}}$$

This is the wave equation, which (anticipating solution) we can write

$$\boxed{\frac{\partial^2 y}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 y}{\partial t^2} \quad \text{with} \quad c = \sqrt{\frac{T}{\rho}}}$$

d'Alembert solution of wave equation

$$\frac{\partial^2 y}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 y}{\partial t^2}$$

y is a function of *x* and *t*.
Define new variables so that
y is now a function of *u* and *v*

$$u = x - ct$$
$$v = x + ct$$

With chain rule
we can show

$$\frac{\partial^2 y}{\partial u \partial v} = 0 \quad \Rightarrow y(u,v) = f(u) + g(v)$$

So general solution of wave equation is

$$\boxed{y(x,t) = f(x-ct) + g(x+ct)}$$

Here *f* and *g* are any functions of *(x-ct)* and *(x+ct)*, determined by initial conditions.   A common scenario is that they are sinusoidal.

Note that $f(x-ct)$ describes f*orward-going* wave and
$g(x+ct)$ describes a *backwards-going* one !

with *c* the (phase)
velocity of the wave

Sinusoidal waves

A very common functional dependence for *f* and *g*...

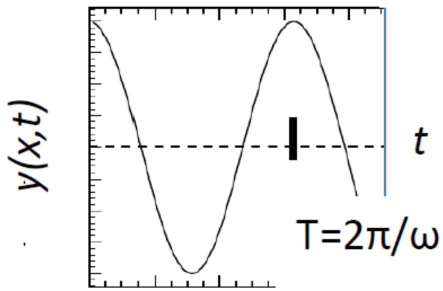
$$y(x,t) = f(x-ct) + g(x+ct)$$

...is sinusoidal. In this case it is usual to write:

$$\boxed{y(x,t) = A\cos(kx-\omega t) + B\cos(kx+\omega t)}$$

with *k* and *ω* (and *A* and *B*) constants

or *Asin(kx-ωt)* ... etc (choice doesn't matter, unless we are comparing one wave with another and then relative phases become important)



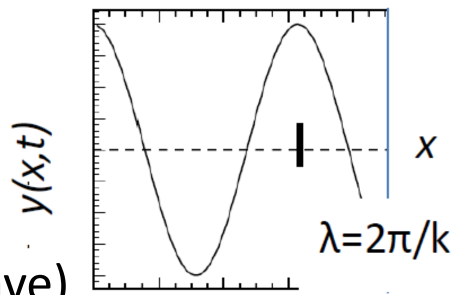
- **speed** of wave $\boxed{c = \omega / k}$
- **frequency** $\boxed{f = 1/T = \omega / 2\pi}$

  where ω is **angular frequency**

- **wavelength** $\boxed{\lambda = 2\pi / k}$

  where k is the **wave-number**

  (or **wave-vector** if also used to indicate direction of wave)



Notation choices

Sinusoidal solution $y(x,t) = A\cos(kx - \omega t)$ (writing here, for compactness, only the forward-going solution)

Using the relationships between *k,ω, λ* & *c* this can be expressed in many forms

$$y(x,t) = A\cos[k(x - ct)]$$

Also note that sometimes it is convenient to write $y(x,t) = A\cos(\omega t - kx)$

A very frequent approach is to use complex notation (we already made use of this when analysing normal modes, and you will have seen it in circuit analysis)

$$y(x,t) = \mathrm{Re}\big[A\exp[i(kx - \omega t)]\big]$$

or $y(x,t) = \mathrm{Im}\big[A\exp[i(kx - \omega t)]\big]$ if it's important to pick out sine function. Note that often the 'Re' or 'Im' is implicit, and it gets omitted in discussion.

Energy and impedance for travelling wave on string

Energy stored in a mechanical wave

Integrate kinetic energy and potential energy densities over an integer number of wavelengths to show they contribute *equally* and give

$$\text{KE density} \equiv \frac{\mathrm{d}K}{\mathrm{d}x} = \frac{1}{2}\rho\left(\frac{\partial y}{\partial t}\right)^2$$

$$\text{PE density} \equiv \frac{\mathrm{d}U}{\mathrm{d}x} = \frac{1}{2}T\left(\frac{\partial y}{\partial x}\right)^2$$

Total energy in *n* wavelengths = $\frac{1}{2}\rho A^2\omega^2 n\lambda$

Power flow = $\frac{1}{2}TA^2\omega k$

Characteristic impedance

The characteristic impedance *Z* is defined as the applied driving force acting in the *y*-direction divided by the velocity of the string in the *y*-direction

$$Z = \frac{F_y}{v_y} = \frac{-T\dfrac{\partial y}{\partial x}}{\dfrac{\partial y}{\partial t}}$$

so with

$$y(x,t) = A\sin(kx - \omega t)$$

$$\Rightarrow Z = \frac{Tk}{\omega} = \frac{T}{c} = \sqrt{(T\rho)}$$

(Note sign on driving force which ensures *Z* positive for forward wave!)

Long Vacation 2011, Q9

9. Two identical long strings are attached to a point mass M. The strings are stretched along the x-axis and are under tension T. The equilibrium position of the mass is at the origin. The mass is now displaced slightly in the transverse direction y and subsequently released. Show that

$$M\left[\frac{\partial^2 y_2}{\partial t^2}\right]_{x=0} = T\left[\frac{\partial y_2}{\partial x} - \frac{\partial y_1}{\partial x}\right]_{x=0},$$

where y_1 and y_2 represent displacements of the string at $x \leq 0$ and $x \geq 0$, respectively. [6]

Show that the amplitude reflection coefficient for a wave incident on the mass is

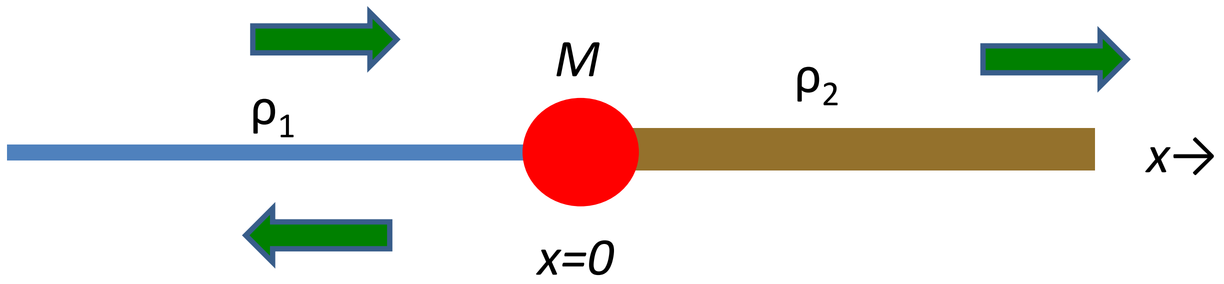
$$r = \frac{-ip}{1+ip}$$

where $p = \frac{\omega^2 M}{2Tk}$, k is the wavenumber $2\pi/\lambda$, and ω is the angular frequency. What is the transmission coefficient? [10]

Sketch the variation of the phase change on reflection as a function of M for fixed ω and T. [4]

Waves at boundaries

This question a good opportunity to remind ourselves what happens to waves at boundaries, so let's answer it in a more general way than is being asked by first allowing strings to be different (*e.g.* different densities)



So we must allow for reflected and transmitted waves. To satisfy boundary conditions all waves must have same frequencies, but their velocity and wave-vector will depend on which string they are on

Incident $\mathrm{Re}\left[A\exp[i(\omega t - k_1 x)]\right]$

Reflected $\mathrm{Re}\left[A'\exp[i(\omega t + k_1 x)]\right]$

Transmitted $\mathrm{Re}\left[A''\exp[i(\omega t - k_2 x)]\right]$

Complex notation convenient for these problems. Pay attention to signs!

Long Vacation 2011, Q9

Write down boundary conditions

1. Equation of motion for mass

$$M\left.\frac{\partial^2 y_{1\,\mathrm{or}\,2}}{\partial t^2}\right|_{x=0} = \left(-T\sin\vartheta_1 - T\sin\theta_2\right)\Big|_{x=0}$$

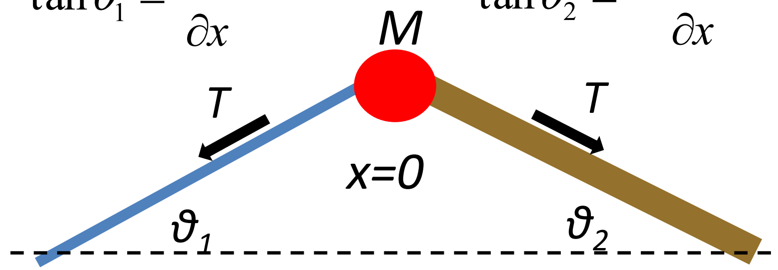
$$\Rightarrow M\left.\frac{\partial^2 y_{1\,\mathrm{or}\,2}}{\partial t^2}\right|_{x=0} \approx T\left.\left(\frac{\partial y_2}{\partial x} - \frac{\partial y_1}{\partial x}\right)\right|_{x=0}$$

2. Strings stay stuck to mass

$$y_1(0,t) = y_2(0,t)$$

$$\sin\theta_1 \approx \tan\theta_1 \qquad \sin\theta_2 \approx \tan\theta_2$$

$$\tan\theta_1 = \frac{\partial y_1}{\partial x} \qquad \tan\theta_2 = -\frac{\partial y_2}{\partial x}$$



Apply to expressions for incident, reflected and transmitted waves

(1)→ $$A''(ik_2T - m\omega^2) = ik_1TA - ik_1TA'$$

(2)→ $$A'' = A + A'$$

Long Vacation 2011, Q9

Evaluate $r \equiv A'/A$ and $t \equiv A''/A$

$$r \equiv \frac{A'}{A} = \frac{T(k_1 - k_2) - im\omega^2}{T(k_1 + k_2) + i\omega^2 m} \qquad t \equiv \frac{A''}{A} = \frac{2k_1 T}{T(k_1 + k_2) + i\omega^2 m}$$

Specialising to case where strings are identical and so $k_1 = k_2 = k$ gives

$$\boxed{r = \frac{-im\omega^2}{2kT + i\omega^2 m} = \frac{-ip}{1+ip} \quad \text{and} \quad t = \frac{2kT}{2kT + i\omega^2 m} = \frac{1}{1+ip} \quad \text{with} \quad p = \frac{m\omega^2}{2kT}}$$

(Aside: one is often asked about transmitted and reflected energy.
So make sure you remember wave power $= \frac{1}{2} T\omega k (\text{Amplitude})^2$)

Phase of r = $-\frac{\pi}{2} - \tan^{-1}(p)$

- m low, phase is $-\pi/2$
- m high, phase is $-\pi$

but this case is slightly artificial, as no reflection in case $m=0$!

Waves - syllabus

Now let's consider

Derivation of the one-dimensional wave equation and its application to transverse waves on a stretched string. D'Alembert's solution. Sinusoidal solutions and their complex representation. Characteristics of wave motion in one dimension: amplitude, phase, frequency, wavelength, wavenumber, phase velocity. Energy in a vibrating string. Travelling waves: energy, power, impedance, reflection and transmission at a boundary. Superposition of two waves of different frequencies: beats and elementary discussion of construction of wave packets; qualitative discussion of dispersive media; group velocity. Method of separation of variables for the one-dimensional wave equation; separation constants. Modes of a string with fixed end points (standing waves): superposition of modes, energy as a sum of mode energies.

with reference to questions: TT 2009, Q6; TT 2010, Q6; TT 2012, Q8

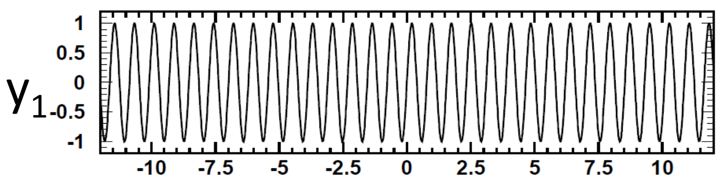
Wave packets and group velocity

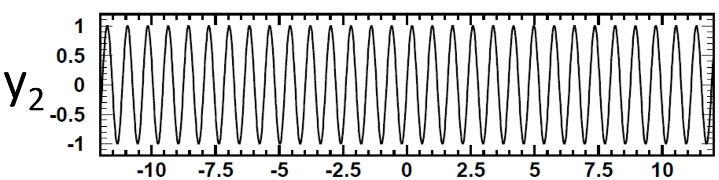
A single wave cannot transmit information. To do that we need a wave packet. Any wave packet can be formed from a sum of single waves. Simplest example:

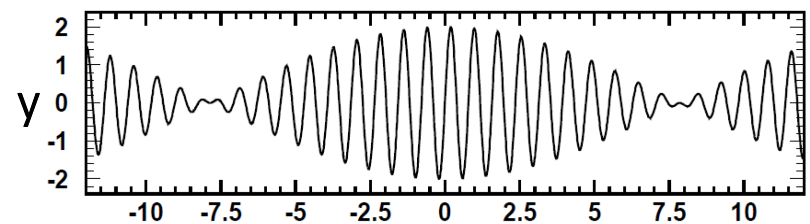
Sum together two waves which differ by 2δω and 2δk in angular frequency and wave-number, respectively:

$$y_1 = A\sin\left[(k+\delta k)x - (\omega+\delta\omega)t\right]$$
$$y_2 = A\sin\left[(k-\delta k)x - (\omega-\delta\omega)t\right]$$

$$\Rightarrow y = y_1 + y_2 = 2A\cos(\delta k\, x - \delta\omega\, t)\sin(kx - \omega t)$$







$$2A\cos(\delta k\, x - \delta\omega\, t)\sin(kx - \omega t)$$

Describes envelope – so envelope moves with velocity $\dfrac{\delta\omega}{\delta k}$ and indeed

Group velocity $v_g = \dfrac{\mathrm{d}\omega}{\mathrm{d}k}$ while phase velocity $v_p = \dfrac{\omega}{k}$

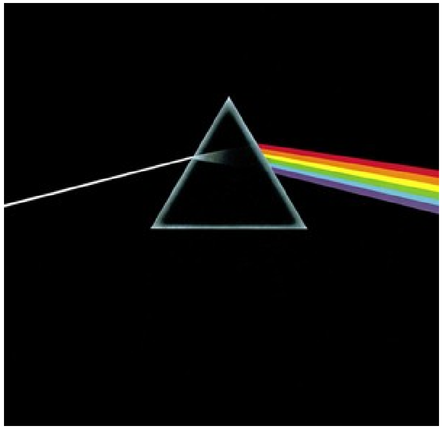
Dispersion

Dispersion is when there is not a linear relationship between *ω* and *k*.

Two consequences:

1. Phase velocity, *ω/k*, depends on *ω* and *k*.
   e.g. Light in medium *m* has refractive index *n* and velocity c_m, where $c_m = c/n$
   That's why a prism splits light.

2. Group velocity ≠ phase velocity



If $v_g = \frac{\mathrm{d}\omega}{\mathrm{d}k}$ and $v_p = \frac{\omega}{k}$ it follows that

$$v_g = v_p + k\frac{dv_p}{dk} \quad \text{and} \quad v_g = v_p - \lambda\frac{dv_p}{d\lambda} \quad \text{and} \quad v_g = \frac{c}{n}\left(1 + \frac{\lambda}{n}\frac{dn}{d\lambda}\right)$$

(more important that you can derive these, rather than learn them!)

Dispersion question: TT 2009, Q6

6. The phase velocity v of light travelling through a gas at a wavelength λ is given by

$$\frac{c^2}{v^2} = A + \frac{B}{\lambda^2} - D\lambda^2 \qquad (1)$$

where A, B, c and D are constants. Show that the group velocity v_g is given by

$$v_g = \frac{v^3}{c^2}(A - 2D\lambda^2)\,.$$

[5]

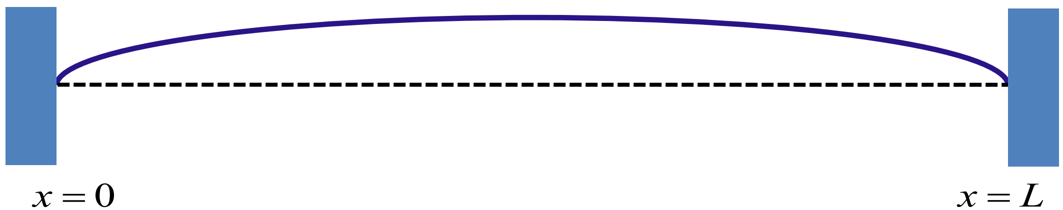
Now $\dfrac{\mathrm{d}\omega}{\mathrm{d}k} = \dfrac{\mathrm{d}\omega}{\mathrm{d}\lambda}\dfrac{\mathrm{d}\lambda}{\mathrm{d}k}$ and $\dfrac{\mathrm{d}\lambda}{\mathrm{d}k} = -\dfrac{2\pi}{k^2}$

and (1) → $c^2 4\pi^2 = A\omega^2\lambda^2 + B\omega^2 - D\omega^2\lambda^4$ so $\dfrac{\mathrm{d}\omega}{\mathrm{d}\lambda} = -\dfrac{\omega}{\lambda}\dfrac{(A-2D\lambda^2)}{c^2}v^2$

hence result

Standing waves: TT 2010, Q6

6. A uniform string of length L which is fixed at both ends has a mass per unit length μ and tension T. Show that transverse standing waves along the string can arise as a result of superposition of two sinusoidal waves travelling in opposite directions. Derive an expression for the displacement of transverse standing waves …



Sum right- and left-going travelling waves of same amplitude and frequency:

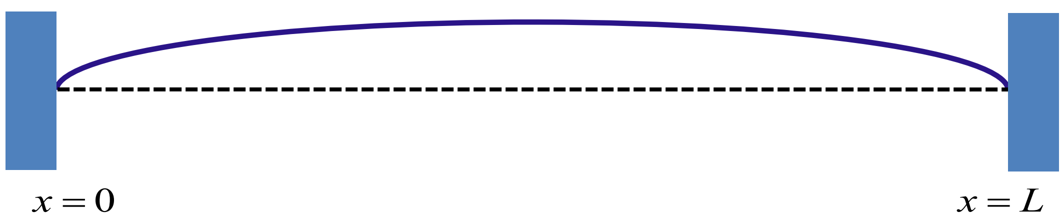
$$
\begin{aligned}
y(x,t) &= A\sin(kx-\omega t) + A\sin(kx+\omega t) \\
&= 2A\sin kx \cos \omega t
\end{aligned}
$$

factorised spatial & temporal dependence

Standing waves: every point on the string moves with a certain time dependence (*cosωt*), but the amplitude depends on its position along the string (*sinkx*)

Standing waves: TT 2010, Q6

… and hence find the frequencies of the normal modes. What is the tension in a violin string of length 325 mm and mass 125 mg, tuned to 660 Hz? [You may assume that the velocity of travelling waves is given by $v = \sqrt{T/\mu}$.]



Ends of string being fixed determine boundary conditions: $y(0,t) = y(L,t) = 0$

$$y(x,t) = 2A\sin kx\cos\omega t \quad \Rightarrow k_n L = n\pi \quad \text{and} \quad \omega_n = n\sqrt{\frac{T}{\mu}}\frac{\pi}{L}$$

angular frequency of normal modes

and putting in numbers, with *n=1*, gives *T=71* N

Wave equation revisited – solving by separation of variables

We have already solved the wave equation using the d'Alembert approach

$$\frac{\partial^2 y}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 y}{\partial t^2}$$

Can also be solved by looking for solutions which have the 'separated' form

$$y(x,t) = X(x)T(t)$$

i.e. that factorise into functions that are separate functions of *x* and *t.*
This is just the situation that applies to standing waves !

$$\frac{\ddot{X}}{X} = \frac{1}{c^2}\frac{\ddot{T}}{T}$$

set each side equal to some separation constant $-k^2$

This yields $X(x) = A\cos kx + B\sin kx$ and $T(t) = D\cos ckt + E\sin ckt$
with *A*,*B*,*D* and *E* constants defined by initial conditions

Standing waves: TT 2012 Q8

8. An elastic, horizontal string with tension T and mass per unit length ρ is held fixed at both ends $x = 0$ and $x = L$. At $t = 0$, the string is displaced transversally along the y direction in such a way that:

$$y(x,0) = \sin\frac{\pi x}{L} - \frac{2}{3}\sin\frac{3\pi x}{L}\,,$$

(a) Calculate the total energy of the string at $t = 0$. [5]

(b) The string is now released and starts to oscillate. Derive the wave equation

$$\frac{\partial^2 y(x,t)}{\partial t^2} - c^2\frac{\partial^2 y(x,t)}{\partial x^2} = 0\,,$$

describing small amplitude transverse waves on the string, and a formula for the wave speed c. [5]

(c) Solve this wave equation to obtain the transverse displacement $y(x,t)$ of the string at time t. [4]

(d) Do you think the string will ever go back to its original $t = 0$ shape? If so, at what time t_1 will it happen for the first time? [2]

(e) Calculate the kinetic energy associated with each standing wave found to be a solution of the wave equation in part (c) as a function of time t. [4]

Standing waves: TT 2012 Q8

Let's rearrange question so we can discuss the relevant topics more clearly

8. An elastic, horizontal string with tension T and mass per unit length ρ is held fixed at both ends $x = 0$ and $x = L$. At $t = 0$, the string is displaced transversally along the y direction in such a way that:

$$y(x,0) = \sin\frac{\pi x}{L} - \frac{2}{3}\sin\frac{3\pi x}{L}\,,$$

Going from general solution to specific solution through applying initial conditions and monitoring subsequent evolution with time

(c) Solve this wave equation to obtain the transverse displacement $y(x,t)$ of the string at time t. [4]

(d) Do you think the string will ever go back to its original $t = 0$ shape? If so, at what time t_1 will it happen for the first time? [2]

Energy of system

(a) Calculate the total energy of the string at $t = 0$. [5]

(e) Calculate the kinetic energy associated with each standing wave found to be a solution of the wave equation in part (c) as a function of time t. [4]

Standing waves: TT 2012 Q8

Know from separation of variables that a solution to wave equation is

$$y(x,t) = (A\cos kx + B\sin kx)(C\cos kct + D\sin kct)$$

and we also have four boundary conditions:

1. String initially at rest, *i.e.* $\partial y/\partial t = 0$ for all *x* $\Rightarrow D = 0$
2. *y(0,t)=0* $\Rightarrow A = 0$
3. *y(L,t)=0* $\Rightarrow kL = n\pi$ where *n* any integer. This is the eigenvalue eqn. and discretises *k*. Each value of *n* corresponds to a normal mode.
4. Form of initial displacement involves normal modes 1 and 3. From these we fix coefficient of mode 1 to be *1* and 3 to be *-2/3*, and all others *0*.

hence $$\boxed{y(x,t) = \sin\frac{\pi x}{L}\cos\frac{\pi ct}{L} - \frac{2}{3}\sin\frac{3\pi x}{L}\cos\frac{3\pi ct}{L}}$$

This first returns to initial displacement when $\boxed{t = 2L/c}$

Energy of standing waves

Normal mode *n* for our string, with given boundary conditions:

$$y_n(x,t) = F_n \sin\frac{n\pi x}{L}\cos\frac{n\pi ct}{L}$$

Calculate kinetic energy, K_n, and potential energy, U_n, for each mode

$$K_n = \int_0^L \frac{1}{2}\rho\left(\frac{\partial y_n}{\partial t}\right)^2 \mathrm{d}x \qquad U_n = \int_0^L \frac{1}{2}T\left(\frac{\partial y_n}{\partial x}\right)^2 \mathrm{d}x$$

Evaluate and sum $\boxed{E_n = K_n + U_n = \frac{\rho L F_n^{\,2}\omega_n^{\,2}}{4}}$ with $\omega_n = n\sqrt{\frac{T}{\rho}}\frac{\pi}{L}$

What about the case when several normal modes are excited (as in question)?

$$\boxed{E = \sum_{n=1}^{\infty} E_n}$$

Note that all cross-terms have vanished due to the orthogonality of sines

i.e. all these terms are zero

$$\int_0^L \sin\frac{n\pi x}{L}\sin\frac{m\pi x}{L}\mathrm{d}x \quad \text{with } n \neq m$$

So total energy is weighted sum of all the excited normal modes

Standing waves: TT 2012 Q8

(a) Calculate the total energy of the string at $t = 0$. [5]

(e) Calculate the kinetic energy associated with each standing wave found to be a solution of the wave equation in part (c) as a function of time t. [4]

Initial energy of string is all in PE

$$U(t=0) = \int_0^L \frac{1}{2} T \left(\frac{\partial y(t=0)}{\partial x} \right)^2 \mathrm{d}x \quad \Rightarrow \boxed{U(t=0) = \frac{5T\pi^2}{4L}}$$

This result makes sense as it equals total energy of system as calculated from

$$E = \sum_{n=1}^{\infty} E_n \quad \text{and} \quad E_n = \frac{\rho L F_n^{\ 2} \omega_n^{\ 2}}{4}$$

Kinetic energy of each standing wave, *i.e.* kinetic energy of each mode

$$K_n = \int_0^L \frac{1}{2} \rho \left(\frac{\partial y_n}{\partial t} \right)^2 \mathrm{d}x \quad \Rightarrow \boxed{K_n = A_n^{\ 2} \frac{\rho}{4L} (n\pi c)^2 \sin^2 \frac{n\pi c t}{L}}$$

where A_n is the amplitude coefficient for mode n (here $A_1=1$ and $A_3=-2/3$ and others $=0$)