Wave Motion

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Practical details

12 lectures, split roughly 1:2 between normal modes and waves

Notes will be handed out every 1-2 weeks.   These are NOT complete,
so you will have to pay attention and follow what I say and write on board!

In particular, the symbol indicates that important information is missing from notes, which you are advised to add yourself during lectures.
Missing material added in this version of slides – hence this symbol is absent.

(Unintentional) mistakes may well feature – there are prizes for spotting these!

Three sets of problem sheets will be distributed. These are inherited from previous lecturers of this course – many thanks to them.

Material will be posted on http://www.physics.ox.ac.uk/users/wilkinsong

Text books

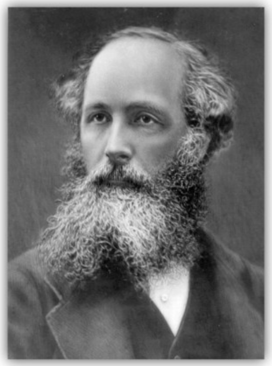
Many excellent books dedicated to oscillations and waves.
Here are three particularly good examples:

- ‘Vibration and waves’, A.P. French, MIT Introductory Physics Series
- ‘Vibration and waves in physics’, I.G. Main, Cambridge University Press
- ‘Waves’, C.A. Coulson and A. Jeffrey, Longman Mathematical Texts

Vibrations and waves in physics

Understanding of both oscillating systems and also wave behaviour is essential for an enormous range of topics across all areas of physics.

Some high profile examples:



Electromagnetic radiation & waves

James Clerk Maxwell

Wave mechanics in quantum theory

Erwin Schrodinger

Ed Witten

String & M-theory:
all ‘fundamental’ particles
are in fact normal modes of oscillating
strings in higher dimensional space
(caution: as yet no experimental evidence!)

Normal modes

Normal modes - introduction

You are familiar with oscillation properties of simple systems with one degree of freedom, *e.g.* simple pendulum. They have a single resonant frequency.

We will here consider systems with more than one degree of freedom (d.o.f.), specifically 2 d.o.f. (and in one dimension), but we will later generalise to *N* d.o.f. when we make the transition to discussing wave motion in continuous media. We will find:

A normal mode of an oscillating system is a pattern of oscillation in which all parts of system move with the same frequency and with a fixed phase relation

We will look at a variety of example systems

e.g. coupled pendula, spring mass systems, double pendula...

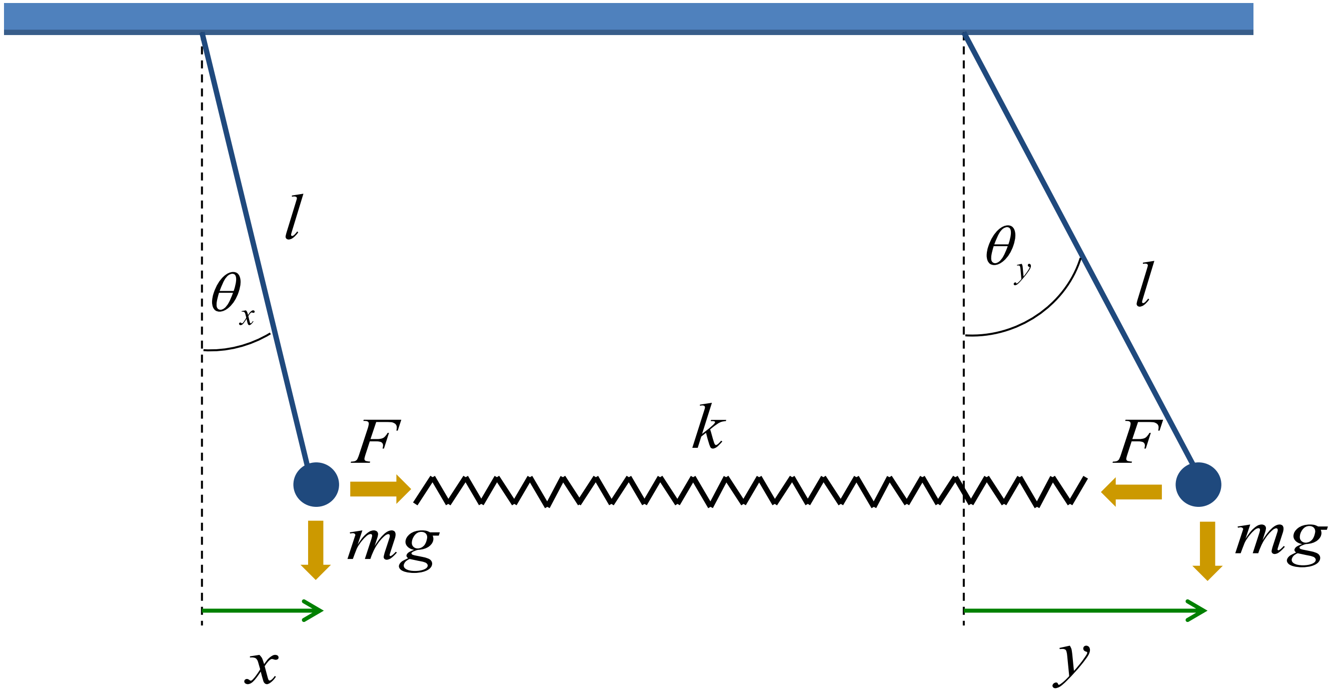
We will explore techniques to find the normal modes, see the dependence on the initial conditions and look at the energies of these systems . We will consider both free motion and motion under an external driving force.

Coupled pendula

We can learn much about normal modes by analysing simple system of two coupled-pendula. We will look at the following:

1. Solving with decoupling method
2. Mode, or normal coordinates
3. The general solution
4. Different initial conditions
5. Energy of system
6. Solving with matrix method

Coupled pendula



Equations of motion:

$$m\ddot{x} = -mg\frac{x}{l} + k(y - x)$$

$$m\ddot{y} = -mg\frac{y}{l} - k(y - x)$$

Solving with decoupling method

Equations of motion:

$$m\ddot{x} = -mg\frac{x}{l} + k(y - x) \quad (1)$$

$$m\ddot{y} = -mg\frac{y}{l} - k(y - x) \quad (2)$$

These are coupled equations, in that both involve the two unknown functions. Lets look for a way to decouple them to facilitate solving.

Adding (1) and (2):

$$m(\ddot{x} + \ddot{y}) = -m\frac{g}{l}(x + y)$$

This is looking familiar... lets define $q_1 \equiv x + y$ then

$$\ddot{q}_1 = -\omega_1^2 q_1 \qquad \text{with } \omega_1^2 = \frac{g}{l}$$

Ah ha! SHM with $\boxed{q_1 = A_1 \cos(\omega_1 t + \varphi_1)}$

where A_1 & ϕ_1 are constants set by boundary conditions

Solving with decoupling method

Equations of motion:

$$m\ddot{x} = -mg\frac{x}{l} + k(y-x) \quad (1)$$

$$m\ddot{y} = -mg\frac{y}{l} - k(y-x) \quad (2)$$

To get other solution subtract (2) from (1):

$$m(\ddot{x}-\ddot{y}) = -m\left(\frac{g}{l} + 2\frac{k}{m}\right)(x-y) \quad (4)$$

Define $q_2 \equiv x - y$ to yield

$$\ddot{q}_2 = -\omega_2^2 q_2 \qquad \text{with } \omega_2^2 = \frac{g}{l} + 2\frac{k}{m}$$

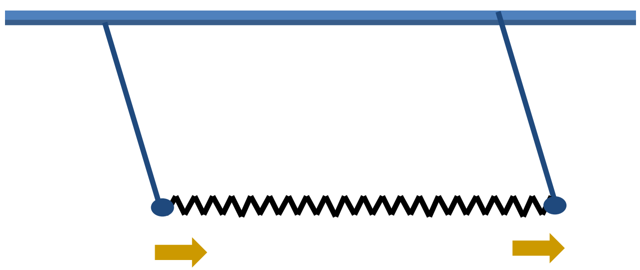
which has solution

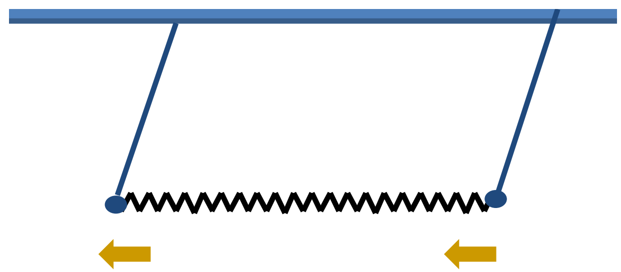
$$\boxed{q_2 = A_2\cos(\omega_2 t + \varphi_2)}$$

Coupled pendula – the normal modes

First normal mode: centre-of-mass motion

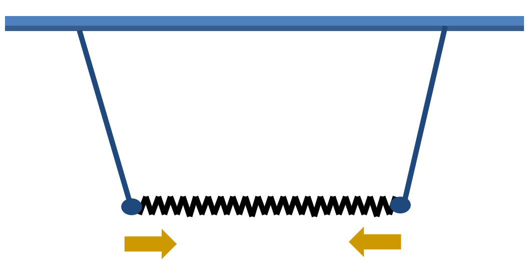
$$q_1 = A_1 \cos(\omega_1 t + \varphi_1) \qquad q_1 \equiv x + y \qquad \omega_1^2 = \frac{g}{l}$$

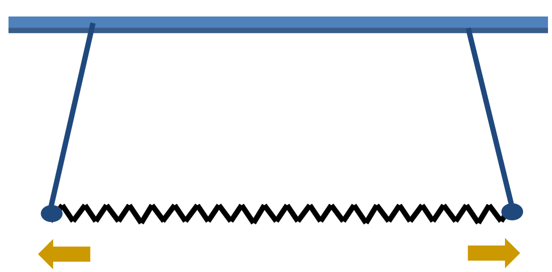




Second normal mode: relative motion

$$q_2 = A_2 \cos(\omega_2 t + \varphi_2) \qquad q_2 \equiv x - y \qquad \omega_2^2 = \frac{g}{l} + 2\frac{k}{m}$$





Mode, or normal, coordinates

The variables q_1 and q_2 are called the mode, or normal, coordinates

In any normal mode only one of these coordinates is active at any one time (*i.e.* either q_1 is vibrating harmonically and q_2 is zero or vice versa)

In fact it is more common to define the mode coordinates with a normalising factor in front (in this case $1/\sqrt{2}$)

$$q_1 \equiv \frac{1}{\sqrt{2}}(x+y)$$

$$q_2 \equiv \frac{1}{\sqrt{2}}(x-y)$$

This means that the vector defined by (q_1,q_2) has same length as that defined by (x,y), *i.e.* $q_1^2 + q_2^2 = x^2 + y^2$. This factor changes none of results we obtained.

Coupled pendula: the general solution

General solution is a sum of the two normal modes

$$x = A_1 \cos(\omega_1 t + \phi_1) + A_2 \cos(\omega_2 t + \phi_2)$$
$$y = A_1 \cos(\omega_1 t + \phi_1) - A_2 \cos(\omega_2 t + \phi_2)$$

The constants A_1 ϕ_1, A_2 and ϕ_2 are set by the initial conditions

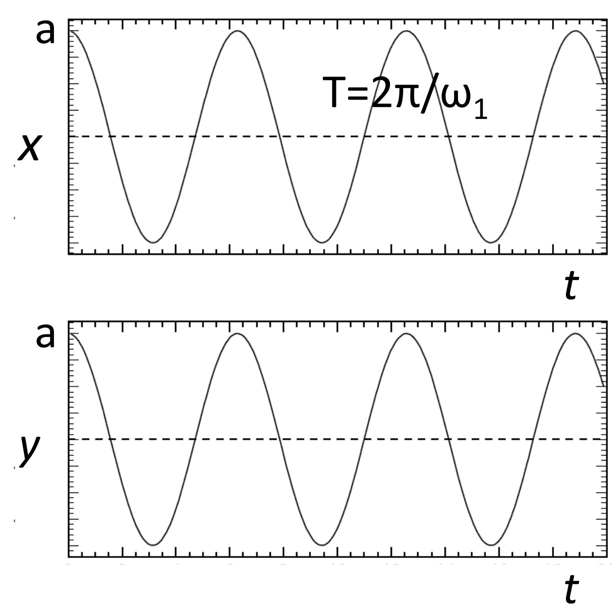
For example

$$x(0) = y(0) = a \quad ; \quad \dot{x}(0) = \dot{y}(0) = 0$$

gives

$$A_1 = a \quad ; \quad A_2 = 0 \quad ; \quad \phi_1 = 0$$

In this case only the 1st normal mode is excited



Coupled pendula: different initial conditions

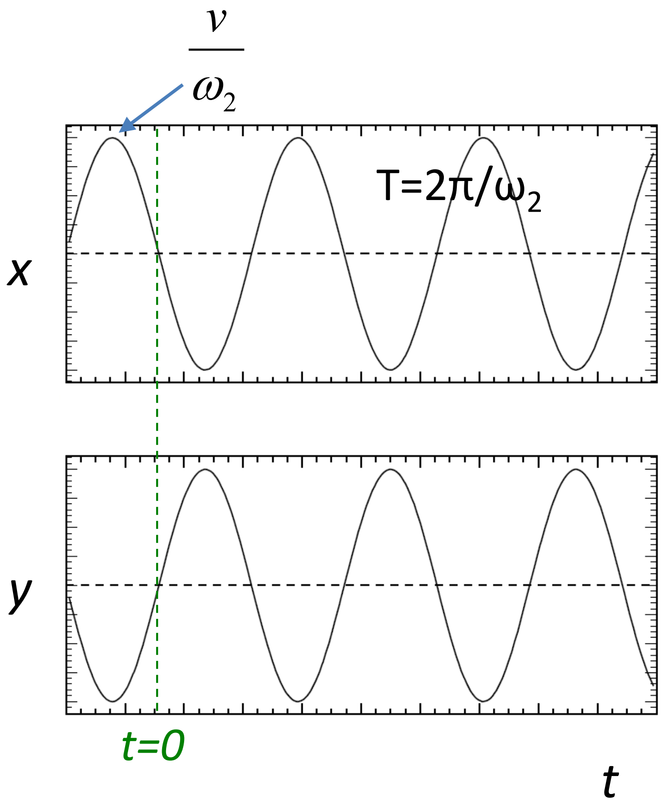
Another example...

The following initial conditions

$$x(0) = y(0) = 0\,; \quad \dot{x}(0) = -v; \quad \dot{y}(0) = v$$

yield

$$A_1 = 0; \quad \phi_2 = \pi/2; \quad A_2 = -\frac{v}{\omega_2}$$



which corresponds to an excitation of the 2nd normal mode

Coupled pendula: different initial conditions

Consider these initial conditions → $x(0)=a; \quad y(0)=0; \quad \dot{x}(0)=\dot{y}(0)=0$

These yield:

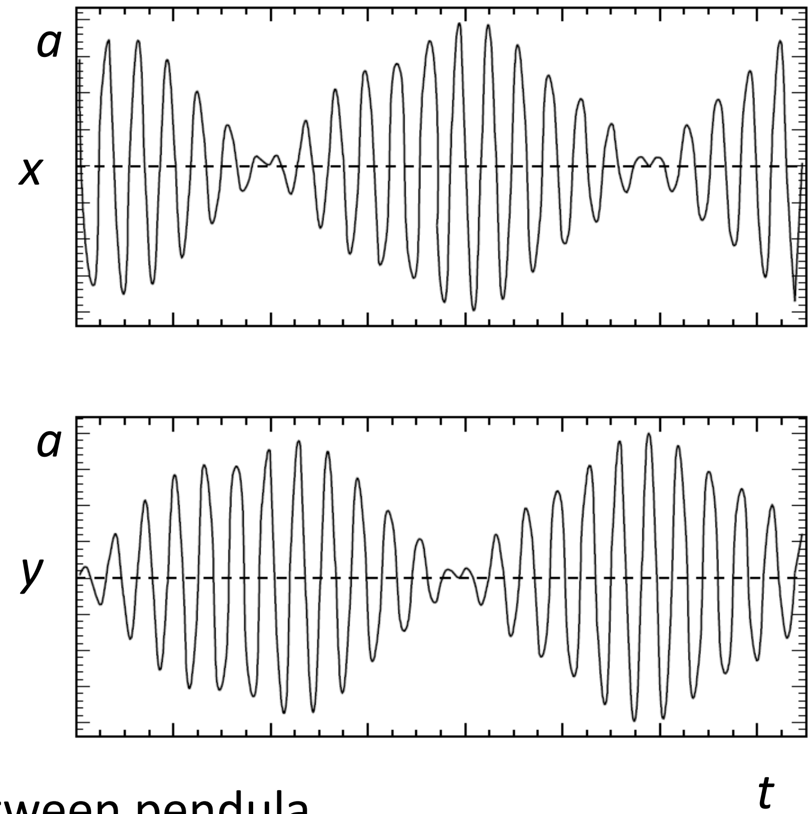
$$A_1 = A_2 = \frac{a}{2}; \quad \phi_1 = \phi_2 = 0$$

Here both normal modes are excited

$$x = a\cos\left(\frac{\omega_1+\omega_2}{2}t\right)\cos\left(\frac{\omega_1-\omega_2}{2}t\right)$$

$$y = -a\sin\left(\frac{\omega_1+\omega_2}{2}t\right)\sin\left(\frac{\omega_1-\omega_2}{2}t\right)$$

Envelope has period $2\pi/[(\omega_1-\omega_2)/2]$



'Beats' – energy is being transferred between pendula

Coupled pendula: different initial conditions

In case it is not obvious how we obtain form of expressions on previous page

Start with

$$x = \frac{a}{2}\left[\cos\omega_1 t + \cos\omega_2 t\right]$$

$$y = \frac{a}{2}\left[\cos\omega_1 t - \cos\omega_2 t\right]$$

Write $S = \dfrac{\omega_1 + \omega_2}{2} t$ and $D = \dfrac{\omega_1 - \omega_2}{2} t$ and so

$$x = \frac{a}{2}\left[\cos(S+D) + \cos(S-D)\right]$$

$$y = \frac{a}{2}\left[\cos(S+D) - \cos(S-D)\right]$$

Then

$$\begin{aligned} x &= \frac{a}{2}\left[\cos(S+D) + \cos(S-D)\right] \\ &= \frac{a}{2}\left[\cos S \cos D - \sin S \sin D + \cos S \cos D + \sin S \sin D\right] \\ &= a \cos S \cos D \end{aligned}$$

$$\begin{aligned} y &= \frac{a}{2}\left[\cos(S+D) - \cos(S-D)\right] \\ &= \frac{a}{2}\left[\cos S \cos D - \sin S \sin D - \cos S \cos D - \sin S \sin D\right] \\ &= -a \sin S \sin D \end{aligned}$$

which is what we want

Energy of coupled pendula

Let's calculate total energy of system, U = T (KE) + V (PE)

Kinetic energy

Trivial ➡ $$T = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}m\dot{y}^2$$

Potential energy

At least two ways to work this out

1) Sum PE in spring $\frac{1}{2}k(y-x)^2$

with PE from gravity $\frac{1}{2}\frac{m}{l}g(x^2+y^2)$ ➡ $$V = \frac{1}{2}m\left(\frac{g}{l}+\frac{k}{m}\right)(x^2+y^2) - kxy$$

2) Recall $F_x = -\frac{\partial V}{\partial x}$ and $F_y = -\frac{\partial V}{\partial y}$

Energy of coupled pendula

<u>Potential energy</u>

2) Recall $F_x = -\dfrac{\partial V}{\partial x}$ and $F_y = -\dfrac{\partial V}{\partial y}$

$$F_x = m\ddot{x} = -mg\frac{x}{l} + k(y-x) = -\frac{\partial V}{\partial x}$$

$$\Rightarrow \quad V(x,y) = mg\frac{x^2}{2l} + \frac{1}{2}kx^2 - kxy + f(y) + C$$

$$F_y = m\ddot{y} = -mg\frac{y}{l} - k(y-x) = -\frac{\partial V}{\partial y}$$

$$\Rightarrow \quad V(x,y) = mg\frac{y^2}{2l} + \frac{1}{2}ky^2 - kxy + f(x) + C$$

and so, neglecting constant, which is an arbitrary offset

$$V(x,y) = \frac{1}{2}m\left(\frac{g}{l} + \frac{k}{m}\right)(x^2 + y^2) - kxy \qquad \text{as before}$$

Energy of coupled pendula

$$U = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2) \quad + \quad \frac{1}{2}m\left(\frac{g}{l} + \frac{k}{m}\right)(x^2 + y^2) - kxy$$

This a bit opaque. How does it look in terms of normal coordinates?

Recall (here with normalisation factors included):

$$x = \frac{1}{\sqrt{2}}(q_1 + q_2)$$
$$y = \frac{1}{\sqrt{2}}(q_1 - q_2)$$

and

$$\omega_1^2 = \frac{g}{l}$$
$$\omega_2^2 = \frac{g}{l} + 2\frac{k}{m}$$

$$\Rightarrow \quad U = \frac{1}{2}m\left(\dot{q}_1^{\ 2} + \ \dot{q}_2^{\ 2}\right) + \frac{1}{2}m\left(\omega_1^{\ 2}q_1^{\ 2} + \ \omega_2^{\ 2}q_2^{\ 2}\right)$$

Much neater – note cross-term in V has disappeared. Indeed:

$$\boxed{U = \underbrace{\left(\frac{1}{2}m\dot{q}_1^{\ 2} + \frac{1}{2}m\omega_1^{\ 2}q_1^{\ 2}\right)}_{\text{energy in mode 1}} + \underbrace{\left(\frac{1}{2}m\dot{q}_2^{\ 2} + \frac{1}{2}m\omega_2^{\ 2}q_2^{\ 2}\right)}_{\text{energy in mode 2}}}$$

So total energy of system = sum of energies in each mode

Solving with matrix method

$$m\ddot{x} = -mg\frac{x}{l} + k(y-x)$$
$$m\ddot{y} = -mg\frac{y}{l} - k(y-x)$$

$\Rightarrow$

$$\begin{pmatrix} \frac{d^2}{dt^2} + \frac{g}{l} + \frac{k}{m} & -\frac{k}{m} \\ -\frac{k}{m} & \frac{d^2}{dt^2} + \frac{g}{l} + \frac{k}{m} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Expecting an oscillatory solution, so let's try substituting one in, making use of complex notation $\Rightarrow$

$$\begin{pmatrix} x \\ y \end{pmatrix} = \mathrm{Re}\begin{pmatrix} X \\ Y \end{pmatrix} e^{i\omega t}$$

X & Y are complex constants

We obtain:

$$\begin{pmatrix} -\omega^2 + \frac{g}{l} + \frac{k}{m} & -\frac{k}{m} \\ -\frac{k}{m} & -\omega^2 + \frac{g}{l} + \frac{k}{m} \end{pmatrix} \begin{pmatrix} X \\ Y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

eigenvector equation

Solving with matrix method

We have an homogeneous matrix equation of the sort $A\Psi = 0$

$$\begin{pmatrix} -\omega^2 + \frac{g}{l} + \frac{k}{m} & -\frac{k}{m} \\ -\frac{k}{m} & -\omega^2 + \frac{g}{l} + \frac{k}{m} \end{pmatrix} \begin{pmatrix} X \\ Y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

eigenvector equation

The non-trivial solution requires the matrix is singular, i.e. has no inverse

$$\Rightarrow \quad \det[A] = 0$$

So here:

$$\begin{vmatrix} -\omega^2 + \frac{g}{l} + \frac{k}{m} & -\frac{k}{m} \\ -\frac{k}{m} & -\omega^2 + \frac{g}{l} + \frac{k}{m} \end{vmatrix} = 0$$

Solving with matrix method

$$\begin{vmatrix} -\omega^2 + \frac{g}{l} + \frac{k}{m} & -\frac{k}{m} \\ -\frac{k}{m} & -\omega^2 + \frac{g}{l} + \frac{k}{m} \end{vmatrix} = 0$$

eigenvalue equation

$$\left(-\omega^2 + \frac{g}{l} + \frac{k}{m}\right) = \pm \frac{k}{m} \quad \longrightarrow \quad \begin{aligned} \omega_1^2 &= \frac{g}{l} \\ \omega_2^2 &= \frac{g}{l} + 2\frac{k}{m} \end{aligned}$$

Substitute back into eigenvector equation to learn

- when $\omega=\omega_1$ then X=Y, call it $A_1 e^{i\phi_1}$ $\quad \Rightarrow x = y = \mathrm{Re}(A_1 e^{i\phi_1} e^{i\omega_1 t})$
$$= A_1 \cos(\omega_1 t + \phi_1)$$
- when $\omega=\omega_2$ then X=-Y, call it $A_2 e^{i\phi_2}$ $\quad \Rightarrow x = -y = \mathrm{Re}(A_1 e^{i\phi_1} e^{i\omega_1 t})$
$$= A_2 \cos(\omega_2 t + \phi_2)$$

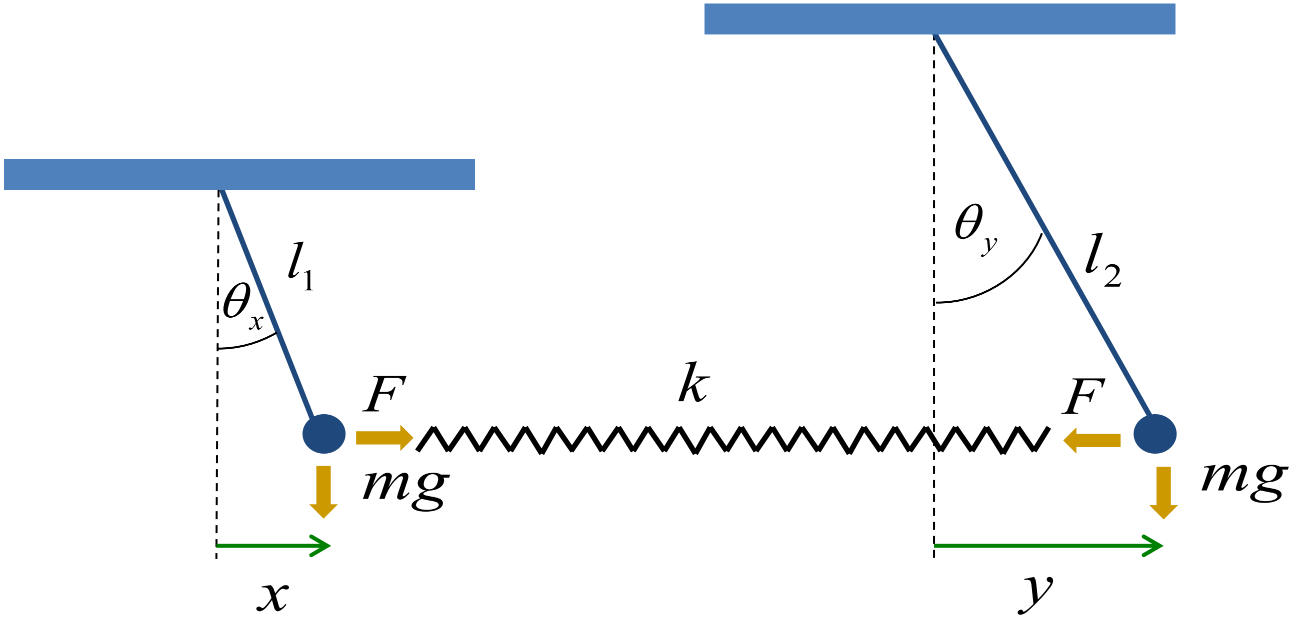
Same normal modes & frequencies as before!

Unequal coupled pendula

Let’s now see if we can solve in the case when the pendula have different lengths

1. Solving with matrix method
2. A specific solution

Unequal coupled pendula



Equations of motion:

$$m\ddot{x} = -mgx/l_1 + k(y-x)$$
$$m\ddot{y} = -mgy/l_2 - k(y-x)$$

Unequal coupled pendula

Attack problem with matrix method:

$$m\ddot{x} = -mgx/l_1 + k(y-x)$$
$$m\ddot{y} = -mgy/l_2 - k(y-x)$$

$$\Longrightarrow \begin{pmatrix} \dfrac{\mathrm{d}^2}{\mathrm{d}t^2} + \dfrac{g}{l_1} + \dfrac{k}{m} & -\dfrac{k}{m} \\ -\dfrac{k}{m} & \dfrac{\mathrm{d}^2}{\mathrm{d}t^2} + \dfrac{g}{l_2} + \dfrac{k}{m} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Try $\begin{pmatrix} x \\ y \end{pmatrix} = \mathrm{Re}\begin{pmatrix} X \\ Y \end{pmatrix} e^{i\omega t}$

$$\Longrightarrow \begin{pmatrix} -\omega^2 + \dfrac{g}{l_1} + \dfrac{k}{m} & -\dfrac{k}{m} \\ -\dfrac{k}{m} & -\omega^2 + \dfrac{g}{l_2} + \dfrac{k}{m} \end{pmatrix} \begin{pmatrix} X \\ Y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

and so we must find solutions of

$$\begin{vmatrix} -\omega^2 + \dfrac{g}{l_1} + \dfrac{k}{m} & -\dfrac{k}{m} \\ -\dfrac{k}{m} & -\omega^2 + \dfrac{g}{l_2} + \dfrac{k}{m} \end{vmatrix} = 0$$

Unequal coupled pendula

Requiring

$$\begin{vmatrix} -\omega^2 + \frac{g}{l_1} + \frac{k}{m} & -\frac{k}{m} \\ -\frac{k}{m} & -\omega^2 + \frac{g}{l_2} + \frac{k}{m} \end{vmatrix} = 0$$

yields $\left(-\omega^2 + \beta_1{}^2 + \frac{k}{m}\right)\left(-\omega^2 + \beta_2{}^2 + \frac{k}{m}\right) = 0$ with $\beta_{1,2}{}^2 = \frac{g}{l_{1,2}}$

Expanding this and then solving for ω^2 gives

$$\boxed{\omega_{1,2}{}^2 = \frac{1}{2}\left[(\beta_1{}^2 + \beta_2{}^2) + \frac{2k}{m} \pm \sqrt{(\beta_1{}^2 - \beta_2{}^2)^2 + \left(\frac{2k}{m}\right)^2}\right]}$$

Sanity check: $l_1 = l_2 = l \Rightarrow \beta_1{}^2 = \beta_2{}^2 = \frac{g}{l}$ and $\omega_{1,2}{}^2$ reduce to equal length solutions

Unequal coupled pendula

Substitute $$\omega_{1,2}{}^2 = \frac{1}{2}\left[(\beta_1{}^2 + \beta_2{}^2) + \frac{2k}{m} \pm \sqrt{(\beta_1{}^2 - \beta_2{}^2)^2 + \left(\frac{2k}{m}\right)^2}\right]$$ with $$\beta_{1,2}{}^2 = \frac{g}{l_{1,2}}$$

into $$\begin{pmatrix} -\omega_{1,2}{}^2 + \frac{g}{l_1} + \frac{k}{m} & -\frac{k}{m} \\ -\frac{k}{m} & -\omega_{1,2}{}^2 + \frac{g}{l_2} + \frac{k}{m} \end{pmatrix}\begin{pmatrix} X \\ Y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

to yield $$\boxed{\left|\frac{Y}{X}\right|_{1,2} = -\frac{2k}{m}\left[(\beta_1{}^2 - \beta_2{}^2) \pm \sqrt{(\beta_1{}^2 - \beta_2{}^2)^2 + (2k/m)^2}\right]^{-1}}$$

In the case l_1=l_2 then $\beta_1{}^2$=$\beta_2{}^2$ and one recovers the same length pendulum solutions *X/Y=+1* and *-1*. It is also interesting to note that one can show

$$\boxed{\left(\frac{Y}{X}\right)_1 = -1/\left(\frac{Y}{X}\right)_2}$$ and so we define $$\boxed{r \equiv \left(\frac{Y}{X}\right)_1 = -\frac{2k}{m}\left[(\beta_1{}^2 - \beta_2{}^2) + \sqrt{(\beta_1{}^2 - \beta_2{}^2)^2 + (2k/m)^2}\right]^{-1}}$$

Unequal coupled pendula: a specific solution

General solution

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ r \end{pmatrix} A_1 \cos(\omega_1 t + \phi_1) + \begin{pmatrix} -r \\ 1 \end{pmatrix} A_2 \cos(\omega_2 t + \phi_2)$$

Now consider the initial conditions

$$x = a;\ y = 0;\ \dot{x} = \dot{y} = 0$$

$$\Rightarrow A_1 = a/(1+r^2);\ A_2 = -ra/(1+r^2);\ \phi_1 = \phi_2 = 0$$

Hence

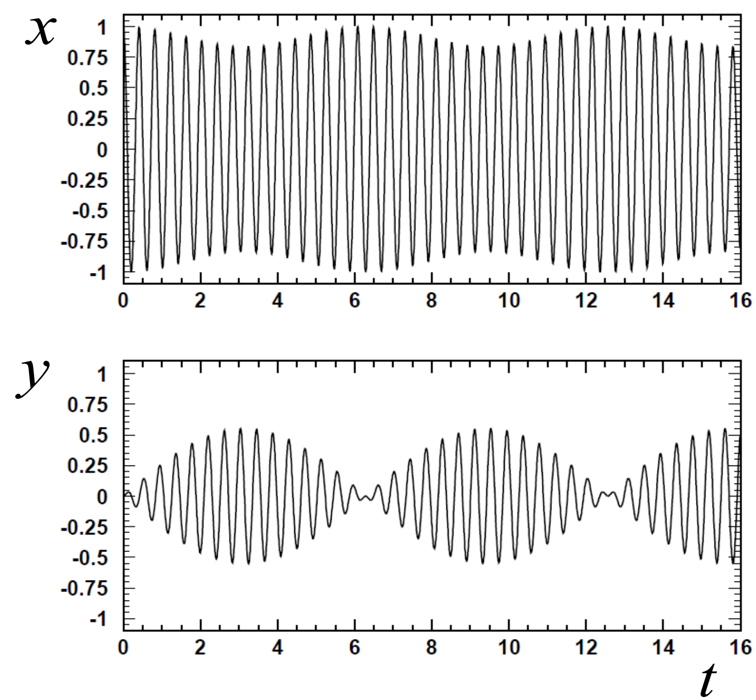
$$x(t) = a\left[\cos\omega_1 t + r^2 \cos\omega_2 t\right]/(1+r^2)$$

$$y(t) = ar\left[\cos\omega_1 t - \cos\omega_2 t\right]/(1+r^2)$$

which can be written

$$x(t) = a\cos\left(\frac{(\omega_1+\omega_2)}{2}t\right)\cos\left(\frac{(\omega_1-\omega_2)}{2}t\right) - a\left(\frac{1-r^2}{1+r^2}\right)\sin\left(\frac{(\omega_1+\omega_2)}{2}t\right)\sin\left(\frac{(\omega_1-\omega_2)t}{2}\right)$$

$$y(t) = -2a\left(\frac{r}{1+r^2}\right)\sin\left(\frac{(\omega_1+\omega_2)}{2}t\right)\sin\left(\frac{(\omega_1-\omega_2)}{2}t\right)$$



'Beats' solution as before, but now with *r< 1* there is incomplete transfer of energy between pendula

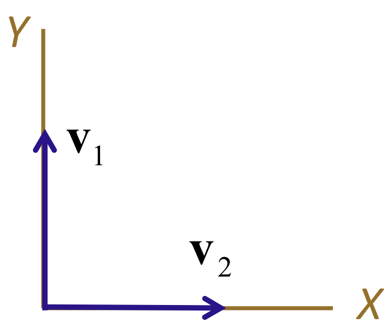
Diagrammatic representation of normal modes

We have seen that systems with 2 d.o.f. have solutions of the sort $\begin{pmatrix} x \\ y \end{pmatrix} = \mathrm{Re}\begin{pmatrix} X \\ Y \end{pmatrix} e^{i\omega t}$ and possess 2 normal modes

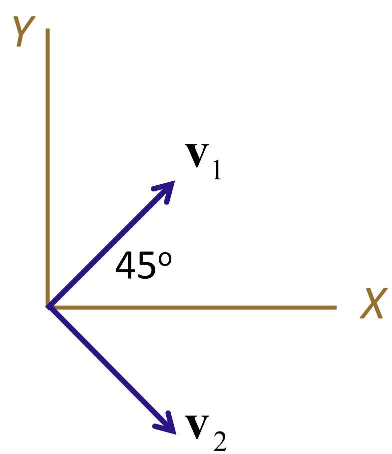
Normal mode motion is specified by the ratio X/Y

Can represent this by a unit-length vector $\mathbf{v} = (X\mathbf{i} + Y\mathbf{j})/\sqrt{X^2+Y^2}$

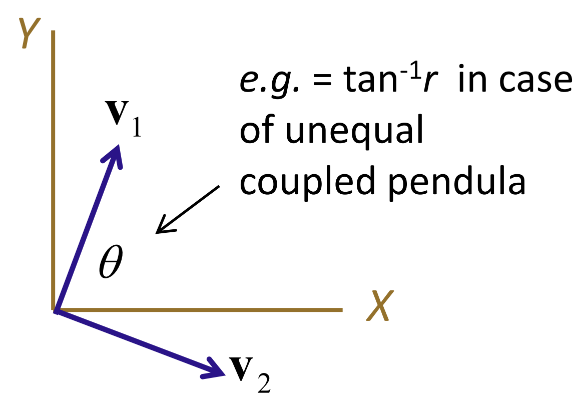
Some examples:



two completely uncoupled pendula



coupled identical pendula



coupled non-identical pendula

Spring-mass systems

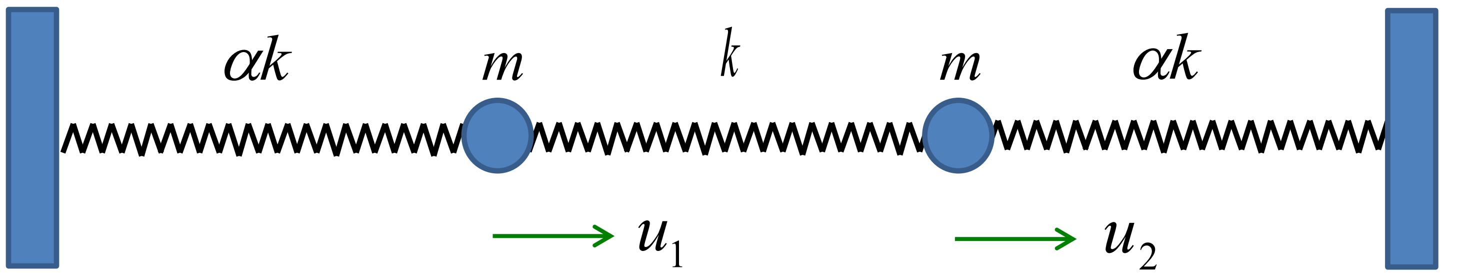
Now we will consider a horizontal system. Same analysis methods will apply as before.

1. Solving with decoupling or matrix method
2. Energy of system
3. A specific solution

And after this we will consider a vertical spring-mass system

Horizontal spring-mass system

Consider two masses moving without friction, between three springs, two with spring constants *αk*, one with spring constant *k*



Equations of motion:

$$m\ddot{u}_1 = -\alpha k u_1 - k(u_1 - u_2)$$
$$m\ddot{u}_2 = -\alpha k u_2 + k(u_1 - u_2)$$

Solutions of horizontal spring-mass system

Equations of motion:

$$m\ddot{u}_1 = -\alpha k u_1 - k(u_1 - u_2)$$
$$m\ddot{u}_2 = -\alpha k u_2 + k(u_1 - u_2)$$

Solve by decoupling method. Write down normal coordinates (here *p* & *q*):

$$p = \frac{1}{\sqrt{2}}(u_1 + u_2) \qquad u_1 = \frac{1}{\sqrt{2}}(p+q)$$
$$q = \frac{1}{\sqrt{2}}(u_1 - u_2) \qquad u_2 = \frac{1}{\sqrt{2}}(p-q)$$

which means

Substituting these in gives:

$$\ddot{p} = -\frac{\alpha k}{m} p \qquad \ddot{q} = -\frac{(\alpha+2)k}{m} q$$

Gives normal frequencies of:

$$\boxed{\omega_p{}^2 = \frac{\alpha k}{m} \qquad \omega_q{}^2 = \frac{(\alpha+2)k}{m}}$$

Motion of two modes is ‘centre-of-mass’ and ‘relative’ as before

Cross-checking with matrix method

Write equations of motion as homogenous matrix equation

$$\begin{pmatrix} \frac{\mathrm{d}^2}{\mathrm{dt}^2}+\frac{\alpha k}{m}+\frac{k}{m} & -\frac{k}{m} \\ -\frac{k}{m} & \frac{\mathrm{d}^2}{\mathrm{dt}^2}+\frac{\alpha k}{m}+\frac{k}{m} \end{pmatrix}\begin{pmatrix} u_1 \\ u_2 \end{pmatrix}=\begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Substitute in the below trial solution

$$\begin{pmatrix} u_1 \\ u_2 \end{pmatrix}=\mathrm{Re}\begin{pmatrix} X \\ Y \end{pmatrix}e^{i\omega t}$$

Demand that the resulting operator matrix is singular, i.e. Det{matrix}=0

$$\begin{vmatrix} -\omega^2+\frac{\alpha k}{m}+\frac{k}{m} & -\frac{k}{m} \\ -\frac{k}{m} & -\omega^2+\frac{\alpha k}{m}+\frac{k}{m} \end{vmatrix}=0$$

Hence get eigenvalue equation

$$\left(-\omega^2+\frac{\alpha k}{m}+\frac{k}{m}\right)^2-\left(\frac{k}{m}\right)^2=0$$

We obtain same solutions as before

$$\boxed{\omega^2=\frac{\alpha k}{m}\ \text{or}\ \frac{(\alpha+2)k}{m}}$$

with

$$\boxed{\begin{matrix} X=Y \\ \text{or} \\ X=-Y \end{matrix}}$$

Energy of horizontal spring-mass system

Let's evaluate total energy, U=K+V, of system

Kinetic energy
$$K = \frac{1}{2}m(\dot{u}_1{}^2 + \dot{u}_2{}^2)$$
$$= \frac{1}{2}m(\dot{p}^2 + \dot{q}^2)$$

Potential energy
$$V = \frac{1}{2}\alpha k u_1{}^2 + \frac{1}{2}k(u_2 - u_1)^2 + \frac{1}{2}\alpha k u_2{}^2$$
$$= \frac{1}{2}\alpha k p^2 + \frac{1}{2}(\alpha + 2)kq^2$$
$$= \frac{1}{2}m\omega_p{}^2 p^2 + \frac{1}{2}m\omega_q{}^2 q^2$$

Total energy

$$\boxed{U = \underbrace{\left(\frac{1}{2}m\dot{p}^2 + \frac{1}{2}m\omega_p{}^2 p^2\right)}_{\text{energy in mode 1}} + \underbrace{\left(\frac{1}{2}m\dot{q}^2 + \frac{1}{2}m\omega_q{}^2 q^2\right)}_{\text{energy in mode 2}}}$$

Again, the sum of energies of each normal mode!

A specific solution for horizontal spring-mass system

General solution is a sum of the two normal modes

$$u_1 = A_p \cos(\omega_p t + \phi_p) + A_q \cos(\omega_q t + \phi_q)$$
$$u_2 = A_p \cos(\omega_p t + \phi_p) - A_q \cos(\omega_q t + \phi_q)$$

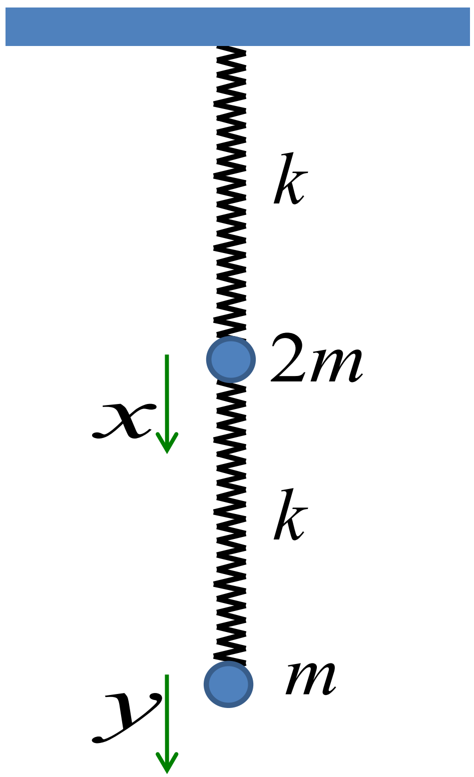
Consider specific initial conditions $u_1 = u_0\,;\; u_2 = 0;\; \dot{u}_1 = \dot{u}_2 = 0$

These imply $A_p = A_q = \dfrac{u_0}{2}\,; \quad \phi_p = \phi_q = 0$

$$\boxed{\begin{aligned} u_1 &= \frac{u_0}{2}\Big(\cos(\omega_p t) + \cos(\omega_q t)\Big) = u_0 \cos\left(\frac{(\omega_p + \omega_q)t}{2}\right)\cos\left(\frac{(\omega_p - \omega_q)t}{2}\right) \\ u_2 &= \frac{u_0}{2}\Big(\cos(\omega_p t) - \cos(\omega_q t)\Big) = -u_0 \sin\left(\frac{(\omega_p + \omega_q)t}{2}\right)\sin\left(\frac{(\omega_p - \omega_q)t}{2}\right) \end{aligned}}$$

Both normal modes excited. The ‘beats’ solution.

Vertical spring-mass system



x and *y* are displacements from equilibrium positions

$$2m\ddot{x} = k(y - x)$$
$$m\ddot{y} = k(x - y)$$

- Find the normal frequencies of the system
- Find the ratio of the amplitudes for each normal mode

Solving with matrix method

Write equations of motion as homogenous matrix equation

$$\begin{pmatrix} \frac{\mathrm{d}^2}{\mathrm{d}t^2} + \frac{k}{m} & -\frac{k}{2m} \\ -\frac{k}{m} & \frac{\mathrm{d}^2}{\mathrm{d}t^2} + \frac{k}{m} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Substitute in the below trial solution

$$\begin{pmatrix} x \\ y \end{pmatrix} = \mathrm{Re}\begin{pmatrix} X \\ Y \end{pmatrix} e^{i\omega t}$$

Demand that the resulting operator matrix is singular, i.e. Det{matrix}=0

$$\begin{vmatrix} -\omega^2 + \frac{k}{m} & -\frac{k}{2m} \\ -\frac{k}{m} & -\omega^2 + \frac{k}{m} \end{vmatrix} = 0$$

Hence get eigenvalue equation

$$\left(-\omega^2 + \frac{k}{m}\right)^2 - \frac{1}{2}\left(\frac{k}{m}\right)^2 = 0$$

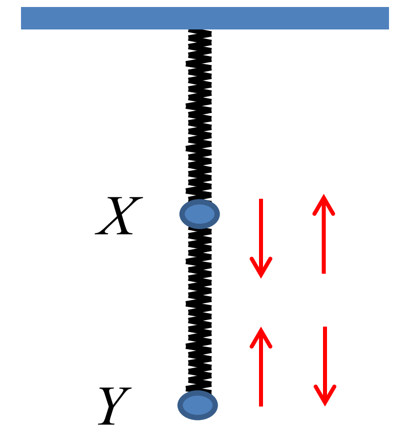
From this we obtain the normal frequencies

$$\boxed{\omega_{1,2}{}^2 = \frac{k}{m}\left(1 \pm \frac{1}{\sqrt{2}}\right)}$$

Normal modes of vertical spring-mass system

Normal mode 1

$$\boxed{\omega_1^{\ 2} = \frac{k}{m}\left(1 + \frac{1}{\sqrt{2}}\right)}$$



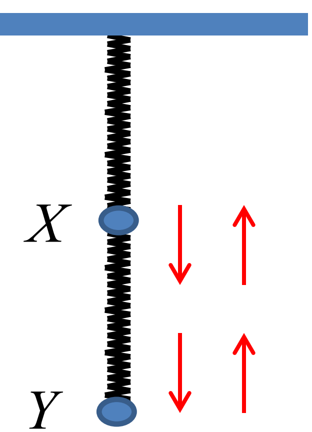
Substitute back into eigenvector equation

$$\left(-\omega_{1,2}^{\ 2} + \frac{k}{m}\right)X - \left(\frac{k}{2m}\right)Y = 0$$

to yield $\boxed{X/Y = -1/\sqrt{2}}$

Normal mode 2

$$\boxed{\omega_2^{\ 2} = \frac{k}{m}\left(1 - \frac{1}{\sqrt{2}}\right)}$$



corresponds to $\boxed{X/Y = 1/\sqrt{2}}$

Coupled oscillators with driving terms

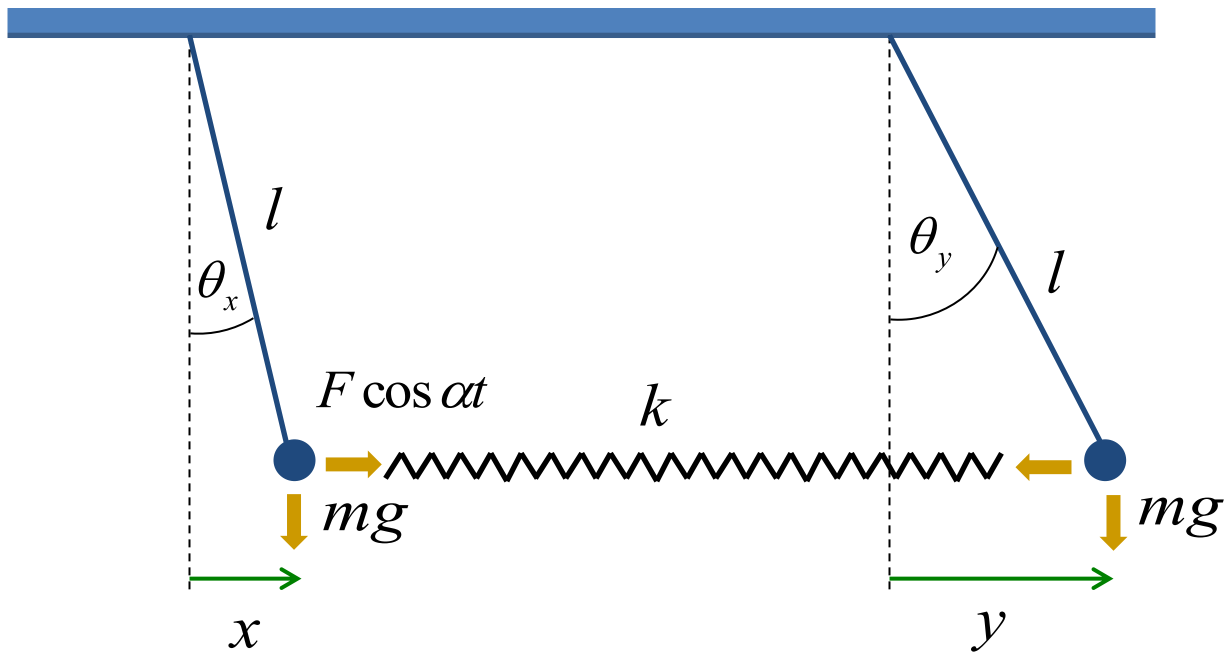
Now we return to our friend the coupled pendula system and analyse it in the case that the oscillations are driven by an external force, and also allow for a damping factor.

1. Finding complementary function
2. Finding particular integral

and finally exercise the same formalism on

3. Horizontal spring mass system

Damped driven coupled pendula



Both pendula experience a retarding force of γ x velocity

Equations of motion:

$$m\ddot{x} = -\gamma\dot{x} - mgx/l + k(y-x) + F\cos\omega t$$

$$m\ddot{y} = -\gamma\dot{y} - mgy/l - k(y-x)$$

Damped driven coupled pendula

Let's arrange equations of motion in form $A\begin{pmatrix} x \\ y \end{pmatrix} = \text{whatever}$

We have:

$$m\ddot{x} = -\gamma\dot{x} - mgx/l + k(y-x) + F\cos\omega t$$
$$m\ddot{y} = -\gamma\dot{y} - mgy/l - k(y-x)$$

and so

$$\begin{pmatrix} \dfrac{\mathrm{d}^2}{\mathrm{d}t^2} + \dfrac{\gamma}{m}\dfrac{\mathrm{d}}{\mathrm{d}t} + \left(\dfrac{g}{l} + \dfrac{k}{m}\right) & -\dfrac{k}{m} \\ -\dfrac{k}{m} & \dfrac{\mathrm{d}^2}{\mathrm{d}t^2} + \dfrac{\gamma}{m}\dfrac{\mathrm{d}}{\mathrm{d}t} + \left(\dfrac{g}{l} + \dfrac{k}{m}\right) \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \frac{F}{m}\begin{pmatrix} 1 \\ 0 \end{pmatrix}\mathrm{Re}[\exp(i\omega t)]$$

Contrary to before, this equation is inhomogeneous, in that RHS≠0.
To solve it we need to find both the complementary function (CF), which is solution to the homogeneous equivalent, and the particular integral (PI)

Damped driven coupled pendula: finding CF

To find CF write down homogenous equation and solve as previously

$$\begin{pmatrix} \frac{\mathrm{d}^2}{\mathrm{d}t^2}+\frac{\gamma}{m}\frac{\mathrm{d}}{\mathrm{d}t}+\left(\frac{g}{l}+\frac{k}{m}\right) & -\frac{k}{m} \\ -\frac{k}{m} & \frac{\mathrm{d}^2}{\mathrm{d}t^2}+\frac{\gamma}{m}\frac{\mathrm{d}}{\mathrm{d}t}+\left(\frac{g}{l}+\frac{k}{m}\right) \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}=0$$

Try $\begin{pmatrix} x \\ y \end{pmatrix}=\mathrm{Re}\begin{pmatrix} X \\ Y \end{pmatrix}e^{i\omega t}$ and find solution when operator matrix is singular

$$\begin{vmatrix} -\omega^2+i\omega\frac{\gamma}{m}+\left(\frac{g}{l}+\frac{k}{m}\right) & -\frac{k}{m} \\ -\frac{k}{m} & -\omega^2+i\omega\frac{\gamma}{m}+\left(\frac{g}{l}+\frac{k}{m}\right) \end{vmatrix}=0$$

Damped driven coupled pendula: finding CF

To find CF write down homogenous equation and solve as previously

$$\begin{vmatrix} -\omega^2 + i\omega\frac{\gamma}{m} + \left(\frac{g}{l} + \frac{k}{m}\right) & -\frac{k}{m} \\ -\frac{k}{m} & -\omega^2 + i\omega\frac{\gamma}{m} + \left(\frac{g}{l} + \frac{k}{m}\right) \end{vmatrix} = 0$$

$$\Rightarrow \left[-\omega^2 + \frac{i\gamma}{m}\omega + \left(\frac{g}{l} + \frac{k}{m}\right)\right]^2 - \left(\frac{k}{m}\right)^2 = 0 \qquad \Rightarrow \left[-\omega^2 + \frac{i\gamma}{m}\omega + \left(\frac{g}{l} + \frac{k}{m}\right)\right] = \pm\left(\frac{k}{m}\right)$$

gives the equations $\quad \bar{\omega}_1{}^2 - \frac{i\gamma}{m}\omega - \frac{g}{l} = 0 \quad$ and $\quad \bar{\omega}_2{}^2 - \frac{i\gamma}{m}\omega - \left(\frac{g}{l} + 2\frac{k}{m}\right) = 0$

with solutions

$$\boxed{\varpi_{1,2} = i\frac{\gamma}{2m} \pm \left(\omega_{1,2}{}^2 - \left(\frac{\gamma}{2m}\right)^2\right)^{\frac{1}{2}}}$$

where $\omega_1^2 = \frac{g}{l}$, $\quad \omega_2^2 = \frac{g}{l} + 2\frac{k}{m}$

which are the results from the undamped scenario

(note, no physical difference between $\pm$ variants. Just use + from now on.)

Damped driven coupled pendula: finding CF

Substitute eigenvalues into the below:

$$\begin{pmatrix} -\varpi_{1,2}{}^2 + \dfrac{i\varpi_{1,2}\gamma}{m} + \left(\dfrac{g}{l}+\dfrac{k}{m}\right) & -\dfrac{k}{m} \\ -\dfrac{k}{m} & -\varpi_{1,2}{}^2 + \dfrac{i\varpi_{1,2}\gamma}{m} + \left(\dfrac{g}{l}+\dfrac{k}{m}\right) \end{pmatrix} \begin{pmatrix} X \\ Y \end{pmatrix} = 0$$

to deduce one mode has X=Y, & the other X=-Y. Since $\begin{pmatrix} x \\ y \end{pmatrix} = \mathrm{Re}\begin{pmatrix} X \\ Y \end{pmatrix} \exp(i\varpi_{1,2}t)$

We get the CF:

$$\boxed{\begin{pmatrix} x \\ y \end{pmatrix} = \exp\left(-\frac{\gamma t}{2m}\right)\left(A_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} \cos\left(\left[\omega_1{}^2 - \left(\frac{\gamma}{2m}\right)^2\right]^{\frac{1}{2}} t + \phi_1 \right) + A_2 \begin{pmatrix} 1 \\ -1 \end{pmatrix} \cos\left(\left[\omega_2{}^2 - \left(\frac{\gamma}{2m}\right)^2\right]^{\frac{1}{2}} t + \phi_2 \right) \right)}$$

Note the exponential decay factor.

Finding CF with decoupling method

The equations of motion

$$m\ddot{x} = -\gamma\dot{x} - mgx/l + k(y-x)$$
$$m\ddot{y} = -\gamma\dot{y} - mgy/l - k(y-x)$$

can be decoupled with the normal coordinates

$$q_1 = \frac{1}{\sqrt{2}}(x+y)$$
$$q_2 = \frac{1}{\sqrt{2}}(x-y)$$

to yield the 2nd order homogeneous differential equations

$$\ddot{q}_1 + \frac{\gamma}{m}\dot{q}_1 + \omega_1{}^2 q_1 = 0 \qquad \ddot{q}_2 + \frac{\gamma}{m}\dot{q}_2 + \omega_2{}^2 q_2 = 0$$

with

$$\omega_1^2 = \frac{g}{l}$$
$$\omega_2^2 = \frac{g}{l} + 2\frac{k}{m}$$

that can be solved through trial solution q=Re($e^{i\omega t}$) to give same results

Damped driven coupled pendula: finding PI

We have the CF. Now we need to find the PI, *i.e*. a solution to the full equation

$$\begin{pmatrix} \frac{\mathrm{d}^2}{\mathrm{dt}^2}+\frac{\gamma}{m}\frac{\mathrm{d}}{\mathrm{dt}}+\left(\frac{g}{l}+\frac{k}{m}\right) & -\frac{k}{m} \\ -\frac{k}{m} & \frac{\mathrm{d}^2}{\mathrm{dt}^2}+\frac{\gamma}{m}\frac{\mathrm{d}}{\mathrm{dt}}+\left(\frac{g}{l}+\frac{k}{m}\right) \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}=\frac{F}{m}\begin{pmatrix} 1 \\ 0 \end{pmatrix}\mathrm{Re}[\exp(i\alpha t)]$$

Try this ansatz $\begin{pmatrix} x \\ y \end{pmatrix}=\mathrm{Re}\left[\begin{pmatrix} P \\ Q \end{pmatrix}e^{i\alpha t}\right]$ which means solving the following

$$\begin{pmatrix} -\alpha^2+i\alpha\frac{\gamma}{m}+\left(\frac{g}{l}+\frac{k}{m}\right) & -\frac{k}{m} \\ -\frac{k}{m} & -\alpha^2+i\alpha\frac{\gamma}{m}+\left(\frac{g}{l}+\frac{k}{m}\right) \end{pmatrix}\begin{pmatrix} P \\ Q \end{pmatrix}=\frac{F}{m}\begin{pmatrix} \frac{1}{0} \end{pmatrix}$$

Damped driven coupled pendula: finding PI

We have matrix equation of the sort: $MU=V$

$$\underbrace{\begin{pmatrix} -\alpha^2 + i\alpha\frac{\gamma}{m} + \left(\frac{g}{l}+\frac{k}{m}\right) & -\frac{k}{m} \\ -\frac{k}{m} & -\alpha^2 + i\alpha\frac{\gamma}{m} + \left(\frac{g}{l}+\frac{k}{m}\right) \end{pmatrix}}_{M} \underbrace{\begin{pmatrix} P \\ Q \end{pmatrix}}_{U} = \underbrace{\frac{F}{m}\begin{pmatrix} 1 \\ 0 \end{pmatrix}}_{V}$$

and so $U=M^{-1}V$, i.e. $\begin{pmatrix} P \\ Q \end{pmatrix} = M^{-1}\frac{F}{m}\begin{pmatrix} 1 \\ 0 \end{pmatrix}$

Damped driven coupled pendula: finding PI

Need to find the inverse of $M = \begin{pmatrix} -\alpha^2 + i\alpha\dfrac{\gamma}{m} + \left(\dfrac{g}{l} + \dfrac{k}{m}\right) & -\dfrac{k}{m} \\ -\dfrac{k}{m} & -\alpha^2 + i\alpha\dfrac{\gamma}{m} + \left(\dfrac{g}{l} + \dfrac{k}{m}\right) \end{pmatrix}$

$$M^{-1} = \frac{1}{\det M}\,\mathrm{adj}(M)$$

$$\begin{aligned}
\det M &= \left[-\alpha^2 + i\alpha\frac{\gamma}{m} + \left(\frac{g}{l} + \frac{k}{m}\right)\right]^2 - \left[\frac{k}{m}\right]^2 \\
&= \left[-\alpha^2 + i\alpha\frac{\gamma}{m} + \left(\frac{g}{l} + \frac{k}{m}\right) - \frac{k}{m}\right]\cdot\left[-\alpha^2 + i\alpha\frac{\gamma}{m} + \left(\frac{g}{l} + \frac{k}{m}\right) + \frac{k}{m}\right] \\
&= \left[-\alpha^2 + \frac{g}{l} + i\alpha\frac{\gamma}{m}\right]\cdot\left[-\alpha^2 + \left(\frac{g}{l} + \frac{2k}{m}\right) + i\alpha\frac{\gamma}{m}\right] \\
&= B_1 e^{-i\theta_1}\cdot B_2 e^{-i\theta_2}
\end{aligned}$$

where $B_{1,2} = \left((\omega_{1,2}{}^2 - \alpha^2)^2 + \left(\dfrac{\alpha\gamma}{m}\right)^2\right)^{\frac{1}{2}}$ and $\tan\theta_{1,2} = \dfrac{-\alpha\gamma/m}{(\omega_{1,2}{}^2 - \alpha^2)}$

Damped driven coupled pendula: finding PI

Now $\operatorname{adj} M = \begin{pmatrix} -\alpha^2 + i\alpha\dfrac{\gamma}{m} + \left(\dfrac{g}{l} + \dfrac{k}{m}\right) & \dfrac{k}{m} \\ \dfrac{k}{m} & -\alpha^2 + i\alpha\dfrac{\gamma}{m} + \left(\dfrac{g}{l} + \dfrac{k}{m}\right) \end{pmatrix}$

and we can write

$$-\alpha^2 + i\frac{\gamma}{m}\alpha + \frac{g}{l} + \frac{k}{m} = \frac{1}{2}\left(-\alpha^2 + i\frac{\gamma}{m}\alpha + \frac{g}{l}\right) + \frac{1}{2}\left(-\alpha^2 + i\frac{\gamma}{m}\alpha + \frac{g}{l} + \frac{2k}{m}\right)$$

$$= \frac{1}{2}\left[B_1 e^{-i\theta_1} + B_2 e^{-i\theta_2}\right]$$

$$\frac{k}{m} = \frac{1}{2}\left(-\alpha^2 + i\frac{\gamma}{m}\alpha + \frac{g}{l} + \frac{2k}{m}\right) - \frac{1}{2}\left(-\alpha^2 + i\frac{\gamma}{m}\alpha + \frac{g}{l}\right)$$

$$= \frac{1}{2}\left[B_2 e^{-i\theta_2} - B_1 e^{-i\theta_1}\right]$$

and so

$$\operatorname{adj} M = \frac{1}{2}\begin{pmatrix} B_1 e^{-i\theta_1} + B_2 e^{-i\theta_2} & B_2 e^{-i\theta_2} - B_1 e^{-i\theta_1} \\ B_2 e^{-i\theta_2} - B_1 e^{-i\theta_1} & B_1 e^{-i\theta_1} + B_2 e^{-i\theta_2} \end{pmatrix}$$

Damped driven coupled pendula: finding PI

$$M^{-1} = \frac{1}{\det M}\,\mathrm{adj}(M) = \frac{e^{i(\theta_1+\theta_2)}}{2B_1B_2}\begin{pmatrix} B_1e^{-i\theta_1}+B_2e^{-i\theta_2} & B_2e^{-i\theta_2}-B_1e^{-i\theta_1} \\ B_2e^{-i\theta_2}-B_1e^{-i\theta_1} & B_1e^{-i\theta_1}+B_2e^{-i\theta_2} \end{pmatrix}$$

$$= \frac{1}{2B_1B_2}\begin{pmatrix} B_1e^{i\theta_2}+B_2e^{i\theta_1} & B_2e^{i\theta_1}-B_1e^{i\theta_2} \\ B_2e^{i\theta_1}-B_1e^{i\theta_2} & B_1e^{i\theta_2}+B_2e^{i\theta_1} \end{pmatrix}$$

Recall $\begin{pmatrix} x \\ y \end{pmatrix} = \frac{F}{m}\mathrm{Re}\left[M^{-1}\begin{pmatrix} 1 \\ 0 \end{pmatrix}e^{i\alpha t}\right]$

$\Rightarrow$ $\boxed{\begin{pmatrix} x \\ y \end{pmatrix} = \frac{F}{2mB_1B_2}\begin{pmatrix} B_1\cos(\alpha t+\theta_2)+B_2\cos(\alpha t+\theta_1) \\ B_2\cos(\alpha t+\theta_1)-B_1\cos(\alpha t+\theta_2) \end{pmatrix}}$ we have it !

with $B_{1,2} = \left((\omega_{1,2}{}^2-\alpha^2)^2+\left(\frac{\alpha\gamma}{m}\right)^2\right)^{\frac{1}{2}}$ and $\tan\theta_{1,2} = \frac{-\alpha\gamma/m}{(\omega_{1,2}{}^2-\alpha^2)}$

Finding PI with decoupling method

Inhomogeneous equations written in terms of normal coordinates

$$\ddot{q}_1 + \frac{\gamma}{m}\dot{q}_1 + \omega_1{}^2 q_1 = \frac{F}{m}\cos\alpha t \quad (1)$$

$$\ddot{q}_2 + \frac{\gamma}{m}\dot{q}_2 + \omega_2{}^2 q_2 = \frac{F}{m}\cos\alpha t \quad (2)$$

$$q_1 = \frac{1}{\sqrt{2}}(x+y) \qquad q_2 = \frac{1}{\sqrt{2}}(x-y)$$

Trial ansatz for (1) $q_1 = \mathrm{Re}\left[A_1 \exp(i\alpha t)\right]$

$$\Rightarrow \left(-\alpha^2 + i\alpha\frac{\gamma}{m} + \frac{g}{l}\right)A_1 = \frac{F}{m}$$

with

$$A_1 = \frac{(F/m)\exp(i\theta_1)}{\left(\left(\omega_1{}^2 - \alpha^2\right)^2 + (\alpha\gamma/m)^2\right)^{1/2}}$$

$$\tan\theta_1 = \frac{-(\alpha\gamma/m)}{(\omega_1{}^2 - \alpha^2)}$$

Hence

$$\boxed{q_1 = \frac{(F/m)}{\left((\omega_1{}^2 - \alpha^2)^2 + \left(\frac{\alpha\gamma}{m}\right)^2\right)^{1/2}}\cos(\alpha t + \theta_1)}$$

Same procedure for (2) gives entirely analogous expression for q_2.
From these same expressions are obtained for *x* and *y* as before.

Damped driven coupled pendula: full solution

Solution = CF + PI

$$\begin{pmatrix} x \\ y \end{pmatrix} = \exp\left(-\frac{\gamma t}{2m}\right)\left(A_1\begin{pmatrix}1\\1\end{pmatrix}\cos\left(\left[\omega_1^{\ 2}-\left(\frac{\gamma}{2m}\right)^2\right]^{\frac{1}{2}}t+\phi_1\right)+A_2\begin{pmatrix}1\\-1\end{pmatrix}\cos\left(\left[\omega_2^{\ 2}-\left(\frac{\gamma}{2m}\right)^2\right]^{\frac{1}{2}}t+\phi_2\right)\right)$$

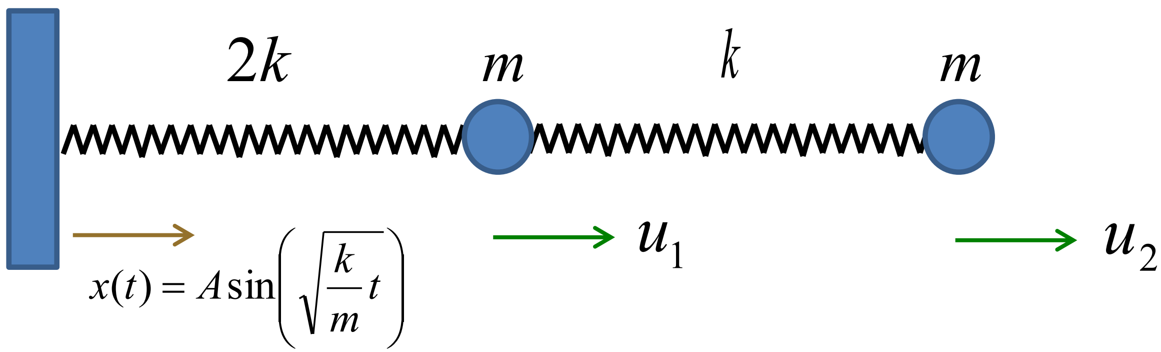
$$+\frac{F}{2mB_1B_2}\begin{pmatrix} B_1\cos(\alpha t+\theta_2)+B_2\cos(\alpha t+\theta_1) \\ B_2\cos(\alpha t+\theta_1)-B_1\cos(\alpha t+\theta_2)\end{pmatrix}$$

with $B_{1,2}=\left((\omega_{1,2}^{\ 2}-\alpha^2)^2+\left(\frac{\alpha\gamma}{m}\right)^2\right)^{\frac{1}{2}}$, $\tan\theta_{1,2}=\frac{-\alpha\gamma/m}{(\omega_{1,2}^{\ 2}-\alpha^2)}$, $\omega_1^2=\frac{g}{l}$ & $\omega_2^2=\frac{g}{l}+2\frac{k}{m}$

The CF part is the ‘transient solution’ determined by the initial conditions; the PI part is the ‘steady state solution’ determined by the driving force.

Horizontal spring-mass system with driving term

Consider two masses moving without friction, with two springs of spring constants *2k* and *k* respectively, connected to wall which is driven by an external force to have time-dependent displacement $x(t) = A\sin\left(\sqrt{\frac{k}{m}}t\right)$



Equations of motion:

$$m\ddot{u}_1 = 2k[x(t) - u_1] - k(u_1 - u_2)$$
$$m\ddot{u}_2 = k(u_1 - u_2)$$

Horizontal spring-mass system with driving term – find the CF

Write down the homogeneous case and find CF using matrix method

$$m\ddot{u}_1 = -3ku_1 + ku_2$$
$$m\ddot{u}_2 = k(u_1 - u_2)$$

Try $\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \mathrm{Re}\begin{pmatrix} U_1 \\ U_2 \end{pmatrix} e^{i\omega t}$ gives $\begin{pmatrix} (-m\omega^2 + 3k) & -k \\ -k & (-m\omega^2 + k) \end{pmatrix} = 0$

Requiring determinant = 0 yields $\boxed{\omega_{1,2}{}^2 = \frac{k}{m}\left[2 \pm \sqrt{2}\right]}$

Substitute back in to eigenvector equ^n $\boxed{\left(\frac{U_2}{U_1}\right)_1 = 1 - \sqrt{2}}$ and $\boxed{\left(\frac{U_2}{U_1}\right)_2 = 1 + \sqrt{2}}$

Horizontal spring-mass system with driving term – find the PI

We have the CF. Now we need to find the PI, *i.e.* a solution to the full equation

$$\begin{pmatrix} \frac{\mathrm{d}^2}{\mathrm{d}t^2}+\frac{3k}{m} & -\frac{k}{m} \\ -\frac{k}{m} & \frac{\mathrm{d}^2}{\mathrm{d}t^2}+\frac{k}{m} \end{pmatrix}\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \frac{Ak}{m}\begin{pmatrix} 2 \\ 0 \end{pmatrix}\mathrm{Re}[\exp(i\sqrt{\frac{k}{m}}t)]$$

Try ansatz

$$\boxed{\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \mathrm{Re}\left[\begin{pmatrix} P \\ Q \end{pmatrix} e^{i\sqrt{\frac{k}{m}}t}\right]}$$

which means solving

$$\begin{pmatrix} \frac{2k}{m} & -\frac{k}{m} \\ -\frac{k}{m} & 0 \end{pmatrix}\begin{pmatrix} P \\ Q \end{pmatrix} = \frac{kA}{m}\begin{pmatrix} 2 \\ 0 \end{pmatrix}$$

Inverse of LHS 2x2 matrix is

$$\frac{1}{-(k/m)^2}\begin{pmatrix} 0 & \frac{k}{m} \\ \frac{k}{m} & \frac{2k}{m} \end{pmatrix} = \frac{m}{k}\begin{pmatrix} 0 & -1 \\ -1 & -2 \end{pmatrix}$$

and so

$$\boxed{\begin{pmatrix} P \\ Q \end{pmatrix} = \begin{pmatrix} 0 \\ -2A \end{pmatrix}}$$

Horizontal spring-mass system with driving term – full solution

Solution = CF + PI

$$\boxed{\begin{aligned}\begin{pmatrix}u_1\\u_2\end{pmatrix} &= A_1\begin{pmatrix}1\\1-\sqrt{2}\end{pmatrix}\cos\left(\left[\frac{k}{m}(2+\sqrt{2}\right]^{\frac{1}{2}}t+\phi_1\right)+A_2\begin{pmatrix}1\\1+\sqrt{2}\end{pmatrix}\cos\left(\left[\frac{k}{m}(2-\sqrt{2}\right]^{\frac{1}{2}}t+\phi_2\right)\\ &\quad+\begin{pmatrix}0\\-2A\end{pmatrix}\cos\sqrt{\frac{k}{m}}t\end{aligned}}$$