NOTE: This problem set is to be handed in to my mail slot labelled SMITH, located in the Clarendon Laboratory by 5:00 PM two days before the scheduled tutorial.

READING: Foot 1.7, 7.1-7.5; Dr Lucas' Quantum Notes

1 Einstein A and B coefficients

Consider a system that consists of atoms with two energy levels E_1 and E_2 and a thermal gas of photons. There are N_1 atoms with energy E_1, N_2 atoms with energy E_2 and the energy density of photons with frequency $\omega = (E_2 - E_1)/\hbar$ is $W(\omega)$. In thermal equilibrium at temperature T, W is given by the Planck distribution:

$$W(\omega) = \frac{\hbar\omega^3}{\pi^2 c^3}\frac{1}{\exp(\hbar\omega/k_B T) - 1}. \tag{1}$$

According to Einstein, this formula can be understood by assuming the following rules for the interaction between the atoms and the photons:

- Atoms with energy E_1 can absorb a photon and make a transition to the excited state with energy E_2; the probability per unit time for this transition to take place is proportional to $W(\omega)$, and therefore given by
$$P_{abs} = B_{12} W(\omega) \tag{2}$$
for some constant B_{12}.
- Atoms with energy E_2 can make a transition to the lower energy state via stimulated emission of a photon. The probability per unit time for this to happen is
$$P_{stim} = B_{21} W(\omega) \tag{3}$$
for some constant B_{21}.
- Atoms with energy E_2 can also fall back into the lower energy state via spontaneous emission. The probability per unit time for spontaneous emission is independent of $W(\omega)$. Lets call this probability
$$P_{spont} = A_{21}. \tag{4}$$

The constants A_{21}, B_{21}, and B_{12} are known as the Einstein A and B coefficients.

(a) Write a differential equation for the time dependence of the occupation numbers N_1 and N_2.

(b) What is the lifetime of the excited energy level E_2 at very low temperature?

(c) Determine the distribution $W(\omega)$ in thermal equilibrium as a function of the Einstein coefficients. You may assume that the ratio N_2/N_1 in thermal equilibrium is given by the Boltzmann factor

$$\frac{N_1}{N_2} = \exp(\hbar\omega/k_B T) \tag{5}$$

(d) By comparing the result of part (c) with the Planck distribution, show that

$$P_{abs} = P_{stim} = \langle n \rangle P_{spont}, \tag{6}$$

where $\langle n \rangle = 1/(exp(\hbar\omega/k_B T) - 1)$ is the average number of photons with frequency ω. Give an interpretation of this formula. When is spontaneous emission dominant? When is stimulated emission dominant?

2 Thermal excitations

A blob of matter is placed in a cavity and allowed to interact with blackbody radiation of temperature T.

(a) Show that for a transition of angular frequency ω_{21}, the rate of stimulated emission becomes equal to that of spontaneous emission when

$$k_B T = \frac{\hbar\omega_{21}}{\ln 2} \tag{7}$$

Below what frequency transition would we expect a significant thermal excitation at room temperature?

(b) Calculate this temperature for the following transitions

i radio frequencies of 50 MHz
ii microwaves at 1 GHz
iii visible light of wavelength 500 nm
iv X-rays of energy 1 keV

For each comment on the implication of your results for excitations of these transitions at room temperature and at the cosmic background temperature. Would you expect thermal excitations for these different transitions in these two cases?

3 Rabi Oscillations

Foot - Problem 7.2

4 Light-matter interaction

(a) Atomic hydrogen is illuminated by light resonant with the $n = 1 \rightarrow n = 2$ Lyman-α transition, linearly polarized along the z-axis. Which upper state(s) can be excited?

(b) Calculate the electric dipole matrix element $\langle 1|ez|2\rangle$ for the transition, expressing your answer in units of ea_0 where a_0 is the Bohr radius. (Look up the relevant hydrogen wavefunctions.)

(c) Use your result to calculate the Einstein A coefficient for the transition, and hence the lifetime of the upper state.

(d) A laser capable of producing continuous wave Lyman-α radiation was recently developed, which yielded a power of 1 nW in a beam of 1 mm diameter. Estimate the Rabi frequency if the laser were tuned to resonance with this transition. Comment on the feasibility of observing Rabi oscillations in this system.

5 Semi-classical two-level atom and the Bloch sphere

Foot - Problem 7.3
Sketch the Bloch sphere representation for part (f).

6 Damping in a two-level atom and steady state

Foot - Problem 7.4