The Train in the Tunnel Problem

1. Example - train in the tunnel.

   A train of proper length $2L = 500\,\mathrm{m}$ approaches a tunnel of proper length $L = 250\,\mathrm{m}$. The trains speed u is such that $g = 2$. An observer at rest with respect to the tunnel measures the trains length to be contracted by a factor of 2 to $250\,\mathrm{m}$ and expects the whole train to fit in the tunnel. An observer on the train knows that the length of the train is $500\,\mathrm{m}$, and that the tunnel is contracted by a factor of 2 to $125\,\mathrm{m}$. Thus the observer on the train argues that the train will not fit into the tunnel. Who is right?

2. Solution

   Two frames, frame S, the rest frame of the tunnel and frame S', the rest frame of the train. An observer in S would assign the position of the left side of the tunnel as $x = 0$ and the right side of the tunnel as $x = L$, while an observer in S' would assign the front of the train as $x' = 0$ and the rear of the train as $x' = -2L$. We check that, as measured in S, the train does indeed fit into the tunnel. Then we need to confirm that, as measured in S', the train does **not** fit into the tunnel. In so doing we can explain this paradox.

   We identify two events, marked by arrows on the viewgraph, and describe them in both frames. Events must be specified by both position and time. The first event is the entry of the train into the tunnel. The clocks in the two frames are set so that at $t = t' = 0$, the front of the train ($x' = 0$) coincides with the left side of the tunnel ($x = 0$), as measured by an observer in S at time $t = 0$ (see viewgraph (a)) and by an observer in S' at time $t' = 0$ (see viewgraph (b)). The second event is the alignment of the right end of the tunnel ($x = L$) and the front of the train ($x' = 0$), as measured by the observer in S (see viewgraph (c)) and the observer in S' (see viewgraph (d)), respectively.

   It follows from $x' = \gamma(x - ut)$ that, for the second event, $0 = \gamma(L - ut)$ so $t = L/u$. This is the time of the second event as seen by the observer in frame S. Now $t' = \gamma(t - ux/c^2)$ so

   $$t' = \gamma\left(\frac{L}{u} - \frac{Lu}{c^2}\right) = \frac{L}{u}\sqrt{1 - \frac{u^2}{c^2}} = \frac{L}{u\gamma} = \frac{t}{\gamma}$$

   This is the time of the second event as measured by the observer in S'. Where, according to the observer in S, is the rear of the train at $t = L/u$? We have

   $$\begin{aligned} x' &= -2L = \gamma\left[x - u\left(\frac{L}{u}\right)\right] = \gamma(x - L) \\ \gamma &= 2 \\ x &= 0 \end{aligned}$$

   Thus at the time measured in S for the second event, the rear of the train is indeed at the tunnel entrance. As observed in S the train fits into the tunnel. By this we mean that the observer in S sees that the front is at the exit of the tunnel and the rear is at the entrance of the tunnel *at the same time*, which in this case is $t = L/u$.

Check that observer in S' does not see the train fit in the tunnel. That observer considers the train to be at rest and the tunnel to be moving towards the train. What is the location in S' of the tunnel entrance at the time of the event specified by the alignment of the front of the train and the tunnel exit—the value of x' when $x = 0$ at time $t' = L/\gamma u$? We have

$$\begin{aligned} x &= \gamma(x' + ut') \\ x &= 0 \\ x' &= -ut' = -L/\gamma = -L/2 \end{aligned}$$

Thus, from the point of view of an observer in S', only one-quarter of the train is in the tunnel when its front reaches the tunnel exit!

Let us make one last check and see in S' the time t' at which the rear of the train is at the tunnel entrance: What is t' when $x' = -2L$ and $x = 0$? We have

$$t' = \gamma \left[\frac{L}{u} - (0) \left(\frac{u}{c^2} \right) \right] = \frac{\gamma L}{u}$$

(not L/u as above).

3. This time is also different from the time $t' = L/u\gamma$ when the front of the train reaches the end of the tunnel. Thus the observer on the train in S' measures the end of the train to pass the tunnel entrance at a later time. The observer in S' will argue that the train's front passes the tunnel exit earlier than the rear passes the tunnel entrance. The differences between the interpretations of the two observers stem from their different notions of simultaneity. These differences allow both observers to be correct in their claims!

4. Assume terrorists build a wall at the end of the tunnel so that the train smashes into it and is brought to rest. Blood and guts, twisted metal and broken glass are everywhere of course, uugh! Let us assume, however, that the train is robust and made of the most rigid material we can imagine, so that when all the dust has cleared away it is essentially undamaged, with the front of the engine at rest against the wall. What will have appeared to have happened, as judged by our two observers? Since from our earlier arguments observers on the ground observe the train entirely in the tunnel for a while, presumably the end of the train does enter the tunnel. However, we also argued that to the train-based observer, the front of the train reached the front of the tunnel before the back enters the tunnel. So presumably in the new circumstances the back will never enter....???? Will the back have actually entered the tunnel or not? Where is the final resting place?

   First, let us clear up what the final situation must be: since both tunnel and train end up at rest with respect to each other, the final position of the back end of the train must be L outside the tunnel! But did it go in and bounce out? Or did it approach the end from outside and then stop?

   We return to our two events. Since there is a frame - the tunnel frame - in which the train is completely in the tunnel, the fact that there is a wall for the train to collide with cannot affect the fact that in that frame, the end of the train and tunnel actually coincided. And since it occurred in that frame, it occurred in everyone else's as well. In the $S-$frame, for example, it occurs at the same instant as the collision in the front.

Can we arrive at a contradiction by saying that in the train's frame we could make the opposite argument; that because the front of our rigid train came to rest before the back end had entered the tunnel, the back end can never do so? This sounds plausible, but does not stand up to closer scrutiny. The fact is that **the back end of the train has no means of finding out that the front has collided until it is already in the tunnel!** So it continues to move at its original speed until that information arrives, which cannot happen faster than the speed of light. By then, it is inside the tunnel, as I shall show below.

We are thus driven to the conclusion that the train really does disappear into the tunnel; that is, every frame of reference records the event where the end of the train passes the entrance of the tunnel. The train must therefore "bounce back" to its final resting place. As the front and back observer can only communicate at light speed, so can the molecules of the train. Thus we are forced to accept that the train really does compress; that observers on the train will notice that the molecules move closer together during the stopping process, eventually returning to their original relative positions (given that the train in fact is not a crumpled mess) after the collision. To illustrate this point consider the following:

What is the time difference between the front of the train hitting the entrance and the rear of the train entering the tunnel in frame S'? From notes this is just

$$\Delta t_1' = \frac{\gamma L}{u} - \frac{L}{u\gamma}$$

but here $\gamma = 2$. So we have

$$\Delta t_1' = \frac{2L}{u} - \frac{L}{2u} = \frac{L}{u}\left[2 - \frac{1}{2}\right]$$

therefore

$$\Delta t_1' = \frac{3L}{2u}$$

Now assume an observer at the front of the train sends a light signal to the rear just as the front collides with the wall. This signal travels $2L$ in the rest frame of the train, and so takes a time

$$\Delta t_2' = \frac{2L}{c}$$

to arrive at the rear of the train. Now for $\gamma = 2$ we need to know u. Thus

$$\begin{aligned} 2 &= \frac{1}{(1 - u^2/c^2)^{1/2}} \\ 4 &= \frac{1}{(1 - u^2/c^2)} \\ 1 - u^2/c^2 &= \frac{1}{4} \end{aligned}$$

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\begin{aligned}
u^2/c^2 &= \frac{3}{4} \\
u &= 0.866c \\
\Delta t'_1 &= (1.5/0.866c)\, L = 1.73L/c \\
\Delta t'_2 &= \frac{2L}{c} \\
\therefore \Delta t'_2 &> \Delta t'_1
\end{aligned}
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therefore the rear of the train is in the tunnel before any signal from the front warning of a collision can reach it. The train must fit in the tunnel! After they all come to rest the train is a length L **out** of the tunnel. The conclusion is the relativity tells us that no material object is perfectly rigid, the train must compress!