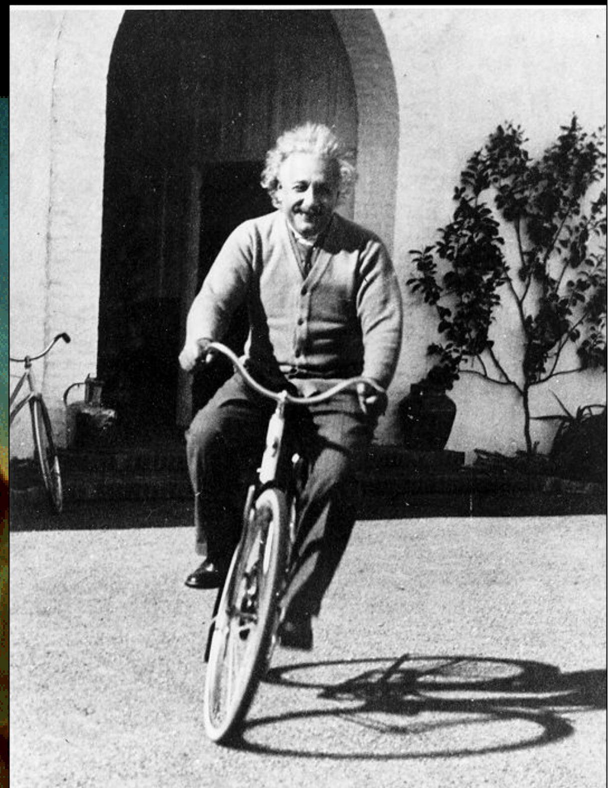
Special Relativity Revision

First Year CP1

Trinity Term

Dr. Robert A. Taylor



What you need to KNOW

The special theory of relativity restricted throughout to problems in one space dimension. The constancy of the speed of light and simultaneity. The Lorentz transformation (derivation not required). Time dilation and length contraction. The addition of velocities. Invariance of the space-time interval. Energy, momentum rest may and their relationship for a single particle. Conservation of energy and momentum. Elementary kinematics of the scattering and decay of sub-atomic particles, including the photon. Relativistic Doppler effect (longitudinal only).

FIRST PUBLIC EXAMINATION

Trinity Term

Preliminary Examination in Physics

CP1 : Mechanics & Special Relativity

also

Moderations in Physics and Philosophy

SPECIMEN PAPER, 9.30 a.m. – 12.00 noon

Time allowed: 2 1/2 hours

Do **not** *turn over the page until told that you may do so.*

Answer **all** *of Section A and* **three** *questions from Section B.*

Start the answer to each question on a fresh page.

A list of physical constants, mathematical formulae and conversion factors accompanies this paper.

The numbers in the margin indicate the weight which the Moderators expect to assign to each part of the question.

Specimen Section A Questions

A7. Starting from the *Lorentz transformation* derive an equation which demonstrates the phenomenon of length contraction. [5]

A8. Write down the relativistic expressions for the total energy E and the momentum $\mathbf{p}$ of a particle of rest mass m in terms of its velocity $\mathbf{v}$ and the speed of light c. Show that $E^2 - (cp)^2 = (mc^2)^2$. A small nuclear power plant in a satellite has a rating of 1 kW, estimate the mass of fuel consumed in 5 years of operation. [6]

Specimen Section B Questions

B4. Show that the quantity $x^2 - (ct)^2$ is a Lorentz invariant. [5]

Show that if a time τ elapses in a space ship moving with velocity $v = c\beta$ relative to the earth, then to an observer on the earth the ship would appear to have moved a distance $l = \gamma\beta c\tau$, where $\gamma = 1/\sqrt{1-\beta^2}$. [5]

Cosmic rays striking the upper atmosphere undergo nuclear collisions with air molecules which result in the creation of μ mesons. The μ mesons are unstable and undergo radioactive decay through the process $\mu \to e + \nu + \bar{\nu}$. In the rest frame of the μ mesons the exponential time constant for decay is $2{\cdot}197\,\mu$s. If the μ mesons are created at a height of 50 km above the earth with a momentum of 10 GeV$/c$ and descend vertically, calculate the fraction which survive to reach the earth's surface. [10]

[The mass of the μ meson is $105{\cdot}7\,\mathrm{MeV}/c^2$.]

Lorentz Transformations

$$x' = \gamma(x - vt)$$
$$y' = y$$
$$z' = z$$
$$t' = \gamma\left[t - \frac{vx}{c^2}\right]$$

Inverse:

$$x = \gamma(x' + vt')$$
$$y = y'$$
$$z = z'$$
$$t = \gamma\left[t' + \frac{vx'}{c^2}\right]$$

where $\gamma = \left[1 - {v^2}/{c^2}\right]^{-\frac{1}{2}}$

and v is the velocity of S' as measured in S

TIME DILATION

In the Lorentz Transformations the interval between 2 events is INVARIANT.

$$\Delta x^2 + \Delta y^2 + \Delta z^2 - c^2 \Delta t^2 = \Delta x'^2 + \Delta y'^2 + \Delta z'^2 - c^2 \Delta t'^2$$

Now for 2 events in S' at the same place (e.g.clock ticks), we have $\Delta x' = 0$. This is the clock rest frame and the time interval between 2 such events is the proper time denoted by $\Delta\tau$. So:

$$c^2 \Delta\tau^2 = c^2 \Delta t^2 - \Delta x^2 - \Delta y^2 - \Delta z^2$$

Divide by Δt^2 then:

$$\left(\frac{\Delta\tau}{\Delta t}\right)^2 = 1 - v^2 / c^2 \qquad \left(\left.\frac{d\mathbf{x}}{dt}\right|_{S'} = \mathbf{v}\right)$$

$$\therefore \Delta\tau = \Delta t \sqrt{1 - v^2 / c^2}$$

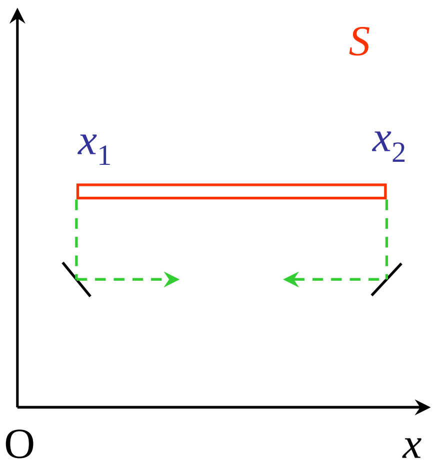
$$\therefore \Delta\tau < \Delta t$$

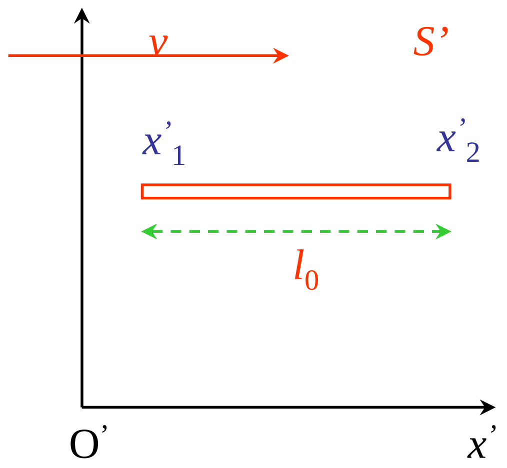
Therefore the time interval is longer than that in the rest frame - TIME DILATION

LORENTZ CONTRACTION

Should be called Lorentz-Fitzgerald contraction

Consider a rigid rod of length l_0 at rest in frame S', moving with velocity v w.r.t. to S.





In S , use light signals as shown to measure x_1 and x_2 at the SAME TIME

In S' use a metre rule to measure l_0.

$$\Delta x' = x_2' - x_1' = l_0$$

From the Lorentz transformations:

$$x_1' = \gamma(x_1 - vt) \qquad x_2' = \gamma(x_2 - vt)$$

$$\Delta x' = \gamma \Delta x \qquad \Delta x = l_0 / \gamma$$

Relativistic Doppler Shift

$$\boxed{\nu' = \nu_0 \left[\frac{1-\beta}{1+\beta}\right]^{\frac{1}{2}}}$$

(Observer moving away from source)

$$\boxed{\nu' = \nu_0 \left[\frac{1+\beta}{1-\beta}\right]^{\frac{1}{2}}}$$

(Observer moving towards source)

Useful way to remember formula:

$$\boxed{\nu' = \nu_0 \gamma (1 \mp \beta)}$$

- sign: moving APART

+ sign: moving TOGETHER

Relativistic Energy

$$W = \int F \, \mathrm{d}x = \int_0^t F v \, \mathrm{d}t$$

Then:

$$Fv = mv^2 \frac{\mathrm{d}\gamma}{\mathrm{d}t} + m\gamma v \frac{\mathrm{d}v}{\mathrm{d}t}$$

$$W = mc^2(\gamma - 1) = mc^2 \left[\frac{1}{\left(1 - v^2/c^2\right)^{\frac{1}{2}}} - 1 \right]$$

This is the **Relativistic Kinetic Energy**.

Then

$$E = W + mc^2 = \gamma mc^2$$

$$\boxed{E^2 - p^2c^2 = m^2c^4}$$

Thus $E^2 - p^2c^2$ is an **INVARIANT**

Transformation of E and p

$$p_x' = \gamma\left(p_x - \beta E/c\right)$$

$$p_y' = p_y$$

$$p_z' = p_z$$

$$E'/c = \gamma\left(E/c - \beta p_x\right)$$

Therefore once more we may define a 4-vector such that:

$$\boxed{X_\mu' = L_{\mu\nu} X_\nu}$$

where X is a 4-vector & $L_{\mu\nu}$ is the Lorentz Transformation matrix.

$$x_\mu = (x,\ y,\ z,\ ict)$$

$$p_\mu = \left(p_x,\ p_y,\ p_z,\ iE/c\right)$$

$$k_\mu = \left(k_x,\ k_y,\ k_z,\ i\omega/c\right)$$

and

$$L = \begin{pmatrix} \gamma & 0 & 0 & i\beta\gamma \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -i\beta\gamma & 0 & 0 & \gamma \end{pmatrix}$$

Relativistic Kinematics

Particle physics units:

m = MeV/c^2 p = MeV/c E = MeV 1 eV = 1.6 x 10^{-19} J

Then

$$\boxed{E^2 = p^2 + m^2}$$

In particle physics $\beta \approx 1$ therefore

$$E = \gamma mc^2 \qquad \& \qquad |\mathbf{p}| = \gamma mc\beta$$

thus

$$\gamma = E/mc^2 \qquad \& \qquad \beta = |\mathbf{p}|c/\mathrm{E}$$

Therefore in particle physics units:

$$\boxed{\gamma = E/m \qquad \& \qquad \beta = |\mathbf{p}|/E}$$ and $$\boxed{t_{\mathrm{part}} = \gamma\tau_{\mathrm{part}} = \frac{E\tau_{\mathrm{part}}}{m}}$$

Centre of Mass or Centre of Momentum

$$S = \left[\sum_i E_i\right]^2 - \left[\sum_i \mathbf{p}_i c\right]^2$$

Is **INVARIANT** for a group of particles.

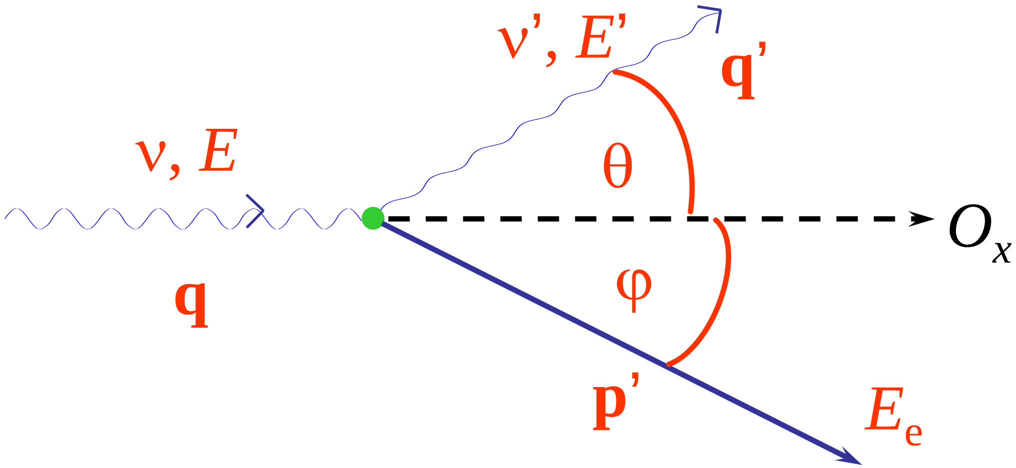
In C of M frame: $$S = \left[\sum_i E_i^*\right]^2 = E_{\mathrm{cm}}^2$$

Where E_i^* is the energy of the i^{th} particle in that frame.

Then:

$$\boxed{\gamma_{\mathrm{cm}} = \frac{\sum_i E_i}{E_{\mathrm{cm}}} \quad \& \quad \beta_{\mathrm{cm}} = \frac{\left|\sum_i \mathbf{p}_i c\right|}{\sum_i E_i}}$$

Compton Scattering



$$E = h\nu, \quad E' = h\nu',$$

$$E_e^2 = (p'c)^2 + \left(mc^2\right)^2$$

Using conservation of energy and momentum we get:

$$\lambda' - \lambda = \frac{h}{mc}(1 - \cos\theta)$$

$$E' = \frac{Emc^2}{E(1 - \cos\theta) + mc^2}$$

$$\cot\varphi = \left(\frac{E}{mc^2} + 1\right)\tan\frac{\theta}{2}$$