Prelims 1997 Question 6

Write down the Lorentz transformations and use them to derive the expressions describing the phenomena of Lorentz contraction and time dilation. [5]

The Prime Minister plans a quiet Cabinet meeting at a remote location. Accordingly, the Cabinet Secretary books a conference on a planet circling a star ten light years from Earth. State, giving reasons for your answers, whether the following statements are in principle TRUE or FALSE. Neglect the effect of acceleration.

(a) The Prime Minister claims that with a fast enough spaceship the journey could be as short as one hour.

(b) The Transport Secretary says that if she leaves 30 minutes after the Prime Minister, but travels at a speed such that her journey time is only 30 minutes, she will arrive at the same time as the Prime Minister.

(c) The Defence Secretary says that the proper time of the Prime Minister's journey will be much longer than an hour.

(d) The Health Minister claims that although the journey might only be one hour, the travellers will age more than ten years.

(e) The Leader of the Opposition says that none of the group will return for at least twenty years. [20]

Answer

(a) TRUE – in PM's frame, the perceived time w.r.t. the Earth/planet frame (the PROPER TIME for the PM) is

$$\Delta\tau_{\mathrm{PM}} = \frac{\Delta t}{\gamma} = \left(1 - \frac{v^2}{c^2}\right)^{\frac{1}{2}} \Delta t$$

where Δt is the Earth/Planet rest frame, which is 10 years in this case. Therefore

$$\begin{aligned} \gamma &= \frac{\Delta t}{\Delta\tau_{\mathrm{PM}}} = \left(\frac{10 \times 365 \times 24}{1}\right) = 87,600 \\ \gamma &= 87,600 \qquad\qquad \beta = 1! \end{aligned}$$

(b) FALSE – (in Earth/planet rest frame). If PM and TS synchronise watches at Heathrow prior to PM leaving, both watches will say the same time when they arrive at planet X. However, the arrival times will be different in planet X's rest frame, as TS has to travel with a $\gamma = 2 \times \gamma_{\mathrm{PM}}$. In fact, TS arrives 30 minutes late as both the PM and the TS travel essentially at c in the Earth/planet rest frame.

(c) FALSE – Be careful with the definition of proper time. If the DS stays behind on Earth he will have to wait a long time before the PM returns. Proper time is the time span measured for a system at rest in a frame, so the problem here is the DS might have meant HIS proper time, at rest in the Earth/planet frame, as opposed to the PM's proper time on HIS watch. You have to be very careful with defintions in special relativity.

(d) FALSE – twins paradox again

(e) TRUE – the round-trip will appear from the Earth to take $\gtrsim$ 10 years, as the ships travel at speeds such that $v \approx c$.

We can calculate the TS's journey time properly for part (b) to show that the common sense answer is, in fact, correct. Distance is x in $S = x/\gamma$ in frame S'. The time for the PM's journey in frame S' is $x/\gamma v = t$. Thus

$$t^2 = \frac{x^2}{v^2}\left(1 - \frac{v^2}{c^2}\right)$$

new speed v_1 needed to halve time is for TS is

$$\frac{t^2}{4} = \frac{x^2}{v_1^2}\left(1 - \frac{v_1^2}{c^2}\right)$$

divide the two equations above to get

$$\begin{aligned}
4 &= \frac{v_1^2}{v^2}\frac{\left(1 - \frac{v^2}{c^2}\right)}{\left(1 - \frac{v_1^2}{c^2}\right)} \\
4v^2\left(1 - \frac{v_1^2}{c^2}\right) &= v_1^2\left(1 - \frac{v^2}{c^2}\right) \\
4v^2 - \frac{4v^2 v_1^2}{c^2} &= v_1^2 - \frac{v_1^2 v^2}{c^2} \\
v_1^2\left(1 - \frac{v^2}{c^2} + \frac{4v^2}{c^2}\right) &= 4v^2 \\
v_1^2 &= \left(\frac{4v^2}{1 + \frac{3v^2}{c^2}}\right)
\end{aligned}$$

Now in frame S these times are x/v and x/v_1. Thus

$$\begin{aligned}
\frac{t(v_1)}{t(v)} &= \frac{x}{v_1}\frac{v}{x} = \frac{v}{(4v^2)^{\frac{1}{2}}}\left(1 + \frac{3v^2}{c^2}\right)^{\frac{1}{2}} \\
&= \frac{1}{2}\left(1 + \frac{3v^2}{c^2}\right)^{\frac{1}{2}} = \frac{1}{2}\left(4 - 3\gamma^{-2}\right)^{\frac{1}{2}} \\
\frac{t(v_1)}{t(v)} &= \frac{1}{2}\left(4 - \sim 4^{-10}\right)^{\frac{1}{2}} = \left(1 - 10^{-10}\right)^{\frac{1}{2}} \\
&= 1 - 5 \times 10^{-11}
\end{aligned}$$

Therefore times in frame S are identical to 1 part in 10^{10}. Thus the transport secretary is $\simeq$ 30 minutes late.