1st Year Relativity - Notes on Lectures 3, 4 & 5

Lecture Three

1. Now lets look at two very important consequences of the LTs, Lorentz-Fitzgerald contraction and time dilation. We'll start with time dilation.

2. From the time dilation diagram: $\Delta\tau$ is the proper time - the time interval in the **REST FRAME** of the object, be it clock, particle, whatever! We get

$$\begin{aligned} \Delta t \text{ (for moving object)} &= \Delta\tau \times \gamma \\ \gamma &\geq 1 \\ \Delta t &\geq \Delta\tau \text{ (time dilation)} \end{aligned}$$

3. Lifetime of particles moving quickly: consider a μ storage ring experiment. If we sample a large number of μ^-, with the same velocity, then in the rest frame the distribution of lifetimes is given by:

$$\frac{dN}{d\tau} = -\frac{N_0}{\tau_m}\, e^{-\tau/\tau_m}$$

where τ_m is the **proper** mean lifetime (or proper lifetime). In the lab, then,

$$\begin{aligned} \tau &= \frac{t}{\gamma} \qquad (\gamma > 1 \text{ and } t > \tau) \\ d\tau &= dt/\gamma \\ \frac{dN}{dt} &= -\frac{N_0}{\gamma\tau_m}\, e^{-t/\gamma\tau_m} \\ \frac{dN}{dt} &= -\frac{N_0}{t_m}\, e^{-t/t_m} \end{aligned}$$

where t_m is the mean lifetime as measured in the **LAB** frame and is related to the proper lifetime by $t_m = \gamma\tau_m$. Therefore we can use the variation of t_m with $\gamma(v)$ to test time dilation in accelerators - see diagram of Bailey experiment from CERN. **N.B. Moving clocks run SLOW**.

4. Lorentz Contraction - from diagram we get $\Delta x = l_0/\gamma$ - contraction. N.B. Measurements must be at the same time in S, whereas you could measure one end of rod at t'_1 then have a cup of tea then measure other end at t'_2. Doesn't matter since rod is at **REST** in S'.

   l_0 is the **PROPER LENGTH** - measured in rest frame of rod.

5. Therefore the length of an object **MEASURED** in the direction of motion is found to be less than the length **MEASURED** when object is at rest. If one looks at, or photographs, a moving body, one does not generally see the precise Lorentz contraction, because what one sees depends upon the light actually reaching the eye (or camera) at that instant, which would have left different parts of the object at different times - e.g. cube — careful analysis of the times of travel from various parts of moving box shows that a single observer photographs the body in a rotated position and with a distorted shape, as in the viewgraph. Another example is that of a moving train, as shown at rest, and moving at $0.9c$ in the figure.

6. Example - Diameter of galaxy is 10^5 light years. How long does it take a proton (in the proton's rest frame) to cross the Galaxy if its energy is 10^{19} eV? $M_p = 938\,\mathrm{MeV/c^2}$. Ignore B fields.

   What are the dimensions of the galaxy in the proton's rest frame? - see diagram. From S,

$$\begin{aligned} t_2 - t_1 &= d/v \\ d &= 10^5 \times 365 \times 24 \times 3600 \times 3 \times 10^8\,\mathrm{m} = 9.46 \times 10^{20}\,\mathrm{m} \end{aligned}$$

   for protons with $E = 10^{19}$ eV ,

$$\begin{aligned} \gamma &= E/m_0c^2 = \frac{10^{19}}{938 \times 10^6} = 1.066 \times 10^{10} \\ v/c &= \left(1 - \frac{1}{\gamma^2}\right)^{\frac{1}{2}} \approx 1 - \frac{1}{2\gamma^2} = 1 - 4.4 \times 10^{-21} \\ \Delta t \text{ in } S &= d/v \approx d/c = 3.15 \times 10^{12}\mathrm{s} \end{aligned}$$

   but in S' the galaxy moves at speed v towards proton and $x'_2 = x'_1 = 0$,

$$\begin{aligned} t_1 &= \gamma(t'_1 + vx'_1/c^2) = \gamma t'_1 \\ t_2 &= \gamma t'_2 \\ \Delta t' &= \Delta t/\gamma \\ \Delta t' &= \frac{3.15 \times 10^{12}}{1.066 \times 10^{10}} = 296\mathrm{s} \end{aligned}$$

   This is the proper time interval between events 1 and 2, measured by one clock at rest with respect to the proton. On the other hand $\Delta t = t_2 - t_1 = 3.15 \times 10^{12}$s in S must be measured by **two** clocks at rest at x_1 and x_2 and synchronised by a light pulse as defined in lecture one. Therefore an observer in S' thinks the Galaxy has a diameter d' given by

$$d' = v\Delta t' = \frac{v\Delta t}{\gamma} = \frac{v}{\gamma} \times \frac{d}{v} = d/\gamma$$

   This is just the Lorentz contraction and so

$$d' = \frac{9.46 \times 10^{20}}{1.066 \times 10^{10}} = 8.875 \times 10^{10}\mathrm{m}$$

   Thus the proton takes 10^5 years in our frame to cross the galaxy but only takes $d'/c = 296\,\mathrm{s}$ in proton's rest frame (5 mins!). In fact d' is less than the earth - sun distance $= 1.5 \times 10^{11}$m.

Lecture Four

7. Meanwhile, back at the interval - lets look once more, and properly this time, at Minkowski space-time diagrams - very useful.

The interval: Consider $\Delta x^2 - c^2\Delta t^2$

$$\begin{aligned}
\Delta x &= \gamma(\Delta x' + v\Delta t') \\
\Delta t &= \gamma(\Delta t' + v\Delta x'/c) \\
\Delta x^2 - c^2\Delta t^2 &= \gamma^2[\Delta x'^2 + 2v\Delta x'\Delta t' + v^2\Delta t'^2 - c^2(\Delta t'^2 + \frac{2v\Delta x'\Delta t'}{c^2} + \frac{v^2\Delta x'^2}{c^4})] \\
&= \gamma^2[\Delta x'^2[1 - v^2/c^2] - c^2\Delta t'^2[1 - v^2/c^2]] \\
(1 - v^2/c^2) &= 1/\gamma^2 \\
\Delta x^2 - c^2\Delta t^2 &= \Delta x'^2 - c^2\Delta t'^2
\end{aligned}$$

8. The interval is often written as

$$\Delta = \Delta x^2 + \Delta y^2 + \Delta z^2 - c^2\Delta t^2$$

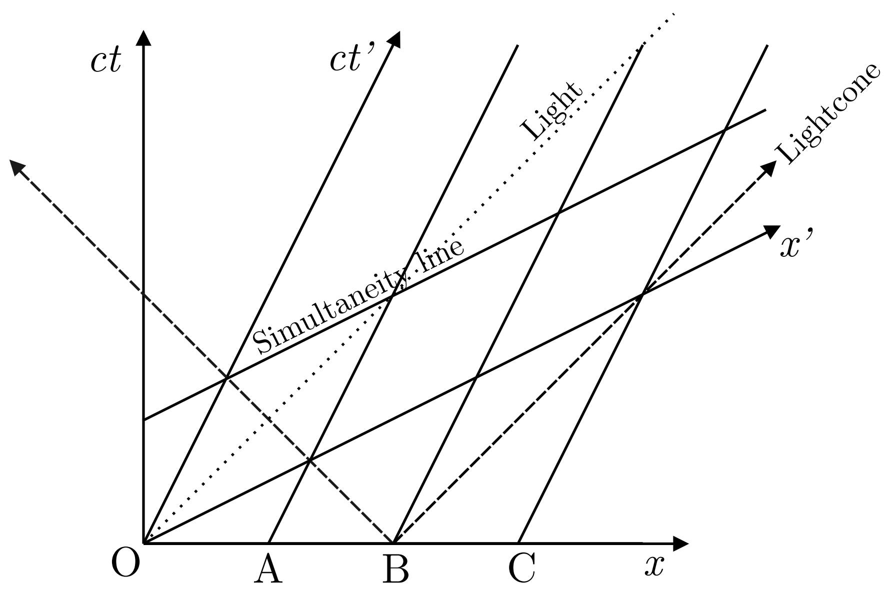
The invariant Δ can be positive, negative or zero. When $\Delta > 0$ we say that the interval is **SPACELIKE** and when $\Delta < 0$ the interval is **TIMELIKE**. Interpret as light cone, $\Delta = 0$ is now. If the interval is spacelike, you can always find a velocity $v < c$ such that $\Delta t' = 0$ (starting in S with given Δx, Δt) but you can **NEVER** find a $v < c$ such that $\Delta x' = 0$.

Conversely if the interval is timelike, you can always find a velocity $v < c$ such that $\Delta x' = 0$ but you can **NEVER** find a $v < c$ such that $\Delta t' = 0$.

Thus two observers in different inertial frames will only agree that two events are simultaneous if they occur at the **SAME** location in space (coalescing into a single event). Two events which are spacelike cannot be linked by light, therefore they cannot **CAUSALLY** affect one another (if the Sun blows up now we've got 8 minutes). Timelike events can be causally related but if Δt is positive then so is $\Delta t'$ — ordering of events does **NOT** change (see diagram).

9. Minkowski Diagram

S' is moving so $\beta = v/c = 1/2$



ct' frame is motion of point $x' = 0$ i.e. since S' is moving at v with respect to S, the line is described by $x = vt$ ($= ct/2$ here). S and S' have $x = x' = t = t' = 0$ (origins coincide at $t = 0$).

What about x'? Connects all points corresponding to $t' = 0$, line of simultaneity in S' — get it by the light cone, and put origin at O as above. N.B. the tilting of the axes is a consequence of representing space-time on a 2D diagram. Any point P represents an EVENT, can be (x, t) or (x', t') and they are **LINEAR** functions of each other. The light signal is at the bisector of all axes. Read off points as shown. N.B. scale is not the same. To get unit distance on each axis we need an invariant! $x^2 - c^2t^2 = 1$. This is the hyperbola in green known as the **CALIBRATION HYPERBOLA.** The diagram shows the symmetry of the Lorentz Transformation.

10. Let us now look at an event in three different inertial frames:

    (a) Rocket — $x = 0$, $ct = 6$ metres

    (b) Lab — From Pythagoras, $ct = 10$ metres and $x = 8$ metres. What is the interval? (a) rocket has $\Delta = -6^2$ m^2 while (b) the lab has $8^2 - 10^2 = -6^2$ m^2.

    (c) Super-rocket — this moves faster that the rocket and thus rocket appears to move backwards. Here by Pythagoras, $x = 20\,\text{m}$, therefore $ct = 20.88\,\text{m}$ $\left(\sqrt{436}\right)$ therefore the interval is $\Delta = 20^2 - 20.88^2 = 400 - 436 = -6^2$!

11. Lorentz Geometry and Worldlines

    The three arrows in the figure are in fact the same - the paper diagram is deceiving us. We have to use the Lorentz geometry of space-time by combining time with space through a **DIFFERENCE** of squares to find the correct magnitude of the result. (space-time vector, the interval)

    *see diagram of worldline, **PRINCIPLE OF MAXIMAL AGING**. Proper time between nearby events along a curved worldline is always $\leq$ time along a direct worldline as measured <u>in that frame</u>. Among all possible worldlines between two events, the straight worldline is unique. All observers agree that it is straight and has the longest proper time — greatest aging, of any possible worldline connecting two events.

12. Causality

    Can a material object move faster than c? No, cause and effect would be violated.

    Example: (see diagram)

    The peace treaty of Shalimar is signed 4 years before the Great Betrayal. The Great Betrayal is the origin (big event). A treaty is signed, the Klingons agree not to attack the Federation in return for data and new technology. Federation ships leave after signing treaty at $0.6c$. Within 4 years, the information given by the Federation allows the Klingons to develop a faster than light projectile — the SUPER! On the day of the G.B. the super is fired at $3c$ towards the departing federation ship. Colonies 1 and 2 lie between the Super and the Federation ship. Both see the passage of the weapon and try to warn the Federation ship — their efforts are in vain since their messages travel at c! At 3 the Super destroys the

Federation ship and a new war is started. What about looking at these events from the Federation ship's point of view?

The G.B. is still at the origin, by definition. Where and when is 3 in this frame?

Klingon frame, $x_3 = 3$ light years, $t_3 = 1$ year, $v = 0.6c$. Therefore

$$\gamma = \frac{1}{(1 - v^2/c^2)^{1/2}} = \frac{1}{(1 - 0.36)^{1/2}} = 1.25$$

From the Lorentz transformation

$$\begin{aligned}
t_3' &= \gamma t_3 - \beta\gamma x_3 && \text{(lost } c \text{ because of units, } t = x/c\text{)}\\
&= 1.25 \times (1 \text{ year}) - 0.6 \times 1.25 \times (3 \text{ years})\\
&= 1.25 - 2.25 = -1 \text{ year!}\\
x_3' &= \gamma x_3 - \beta\gamma t_3 && \text{(units once more!)}\\
&= 1.25 \times (3 \text{ years}) - 0.6 \times 1.25 \times (1 \text{ year})\\
&= 3.75 - 0.75 \text{ years} = 3 \text{ years}
\end{aligned}$$

Therefore we plot event 3 as shown in the diagram. Now connect to the origin and place 1 and 2 in the correct position. Similarly for the signing at Shalimar,

$$\begin{aligned}
x_{\text{sh}} &= 0 \text{ , } t_{\text{sh}} = -4 \text{ years (lab/Klingon co-ordinates)}\\
t_{\text{sh}}' &= \gamma t_{\text{sh}} - \beta\gamma x_{\text{sh}} = -5 \text{ years}\\
x_{\text{sh}}' &= \gamma x_{\text{sh}} - \beta\gamma t_{\text{sh}} = +3 \text{ years}
\end{aligned}$$

as marked. Therefore the worldline of the Federation ship is vertical, and of the Klingons in the diagram is as shown. For the Federation, $t_3' = -1$ year, i.e. **BEFORE** the G.B.! The diagram says that the Super moves **FROM** the Federation ship **TO** the Klingons, but the Federation ship is destroyed not the Klingons!

13. Space travel — is it possible?

    99 light years from earth is Canopus. We are asked to go there, photograph it and return home. But that's impossible we say, only 40 years or so is left of our lives and the round trip is 198 years! Assume that our ship can travel at a speed of $\frac{99}{101}c$ relative to the Earth. We can therefore travel 99 light years in 101 years in Earth's frame ($v = 0.9802c$). The whole trip will therefore take 202 years in Earth's frame. What about rocket time (rocket's inertial frame). Rocket time is proper time

$$\begin{aligned}
\Delta\tau &= \Delta t/\gamma\\
\Delta\tau &= 101/\gamma\\
\& \ \gamma &= \frac{1}{\left(1 - (0.9802)^2\right)^{1/2}} = 5.05\\
\Delta\tau &= 20 \text{ years!}
\end{aligned}$$

Or we can use the invariance of the interval

$$(\Delta x_{\rm rkt})^2 - (c\Delta t_{\rm rkt})^2 = (\Delta x_{\rm earth})^2 - (c\Delta t_{\rm earth})^2$$

but $\Delta x_{\rm rkt} = 0$, therefore

$$\begin{aligned} -\Delta t_{\rm rkt}^2 &= \left(\frac{\Delta x_{\rm earth}}{c}\right)^2 - \Delta t_{\rm earth}^2 \\ &= 99^2 - 101^2 = -20^2 \text{ years}^2 \\ \Delta t_{\rm rkt} &= 20 \text{ years} \end{aligned}$$

thus space travel is possible on paper.

14. Rocket problems

Can we build a rocket to travel at $0.5c$? Initial mass of rocket is m_i, gases are emitted at a speed w relative to the rocket until the rocket's speed reaches u when the final rocket mass is m_f (problem 9.4 in Rosser). This gives

$$\frac{m_i}{m_f} = \left(\frac{c+u}{c-u}\right)^{c/2w}$$

If we take current rocket technology we find that $w \sim 10$ kms^{-1}. Therefore

$$\frac{c}{2w} = \frac{3\times 10^8}{2\times 10^4} = 1.5\times 10^4$$

for $u = \frac{1}{2}c$ we get

$$\frac{m_i}{m_f} = \left(\frac{c+\frac{1}{2}c}{c-\frac{1}{2}c}\right)^{1.5\times 10^4} = 3^{1.5\times 10^4} \approx 10^{7157}!!!$$

We would need a spaceship with an initial mass of 10^{7157} tonnes to end up with 1 tonne!

What about a photon drive or a particle drive? Then $w \approx c$ and $\frac{m_i}{m_f} = \sqrt{3}$. Other small problems that would be encountered are things like the energy of collision with a 1gram dust particle. At 0.5c this could destroy the ship (K.E. $= mc^2(\gamma - 1)$, where $\gamma = 1.55$, giving 14TJ, which is equivalent to a mass of 1000 Kg travelling at 118 kms^{-1}!)

Lecture Five

15. Transformation of Velocity - Consider differences between two events, (x_1, t_1) and (x_2, t_2) in one inertial frame and the **SAME** two events (x_1', t_1') and (x_2', t_2') in another inertial frame. Then

$$\begin{aligned} \Delta x &= \gamma(\Delta x' + v_{\rm rel}\Delta t') \\ \Delta t &= \gamma(\Delta t' + \frac{v_{\rm rel}\Delta x'}{c^2}) \end{aligned}$$

N.B. $v_{\rm rel}$ is the speed of one frame **relative** to the other. We get, on division of the above two equations:

$$\begin{aligned} \frac{\Delta x}{\Delta t} &= \frac{\Delta x' + v_{\rm rel}\Delta t'}{\Delta t' + v_{\rm rel}\Delta x'/c^2} \\ &= \frac{\Delta x'/\Delta t' + v_{\rm rel}}{1 + \frac{v_{\rm rel}}{c^2}\frac{\Delta x'}{\Delta t'}} \\ v &= \frac{v' + v_{\rm rel}}{1 + v' v_{\rm rel}/c^2} \end{aligned}$$

This is the Law of Combination of Velocities and we get:

$$L(v_1)L(v_2) = L\left(\frac{v_1 + v_2}{1 + v_1 v_2/c^2}\right)$$

16. What happens when v_1 and $v_2 \ll c$? **THEN** $v = v_1 + v_2$!

    what happens as v_1 and $v_2 \to c$?

$$v = \frac{c + c}{1 + \frac{c^2}{c^2}} = \frac{2c}{2} = c$$

17. Fizeau's Experiment - see diagram

$$\begin{aligned} u'_x &= \frac{c}{n'} \\ u'_y &= 0 \\ u'_z &= 0 \\ u_x &= \frac{u'_x + v}{1 + v u'_x/c^2} \\ &= \frac{c/n' + v}{1 + vc/n'c^2} \\ &= \frac{c}{n'}\left(1 + \frac{n'v}{c}\right)\left(1 + \frac{v}{n'c}\right)^{-1} \\ &\approx \frac{c}{n'}\left(1 + \frac{n'v}{c}\right)\left(1 - \frac{v}{n'c}\right) \\ &\approx \frac{c}{n'} + v\left(1 - \frac{1}{n'^2}\right) \\ &= \frac{c}{n'} + fv \end{aligned}$$

If Galileo was right,

$$u_x = u'_x + v = \frac{c}{n'} + v$$

and we would have $f = 1$. Michelson and Morley in 1896 did the experiment and measured

$$f = 0.434 \pm 0.020$$

for $n' = 1.33$ (water)

$$f = 1 - \frac{1}{n'^2} = 1 - \frac{1}{1.33^2} = 0.435$$

18. Doppler Effect

Consider a periodically flashing light that moves at speed u toward an observer (diagram). Source is at origin in S'. Suppose one pulse is emitted every τ' in S', $\nu_0 = 1/\tau'$ is the frequency of emission in the rest frame of the source. Observer on right in S sees S' moving towards him at speed u. He measures the time between the arrival of the first and $(N+1)^{\text{th}}$ wave as t. He sees N waves fitted into a distance ct diminished by ut (movement of source). Thus λ is given by:

$$\lambda = \frac{\text{distance}}{\text{No.of waves}} = \frac{ct - ut}{N} = \frac{(c-u)t}{N}$$

Therefore the frequency measured by an observer in S is:

$$\nu_1 = \frac{c}{\lambda} = \frac{cN}{(c-u)t} = \frac{N}{(1-u/c)}\frac{1}{t}$$

But using the Lorentz transformation gives time dilation of the form:

$$\begin{aligned} \frac{\tau}{\gamma} &= \tau' \\ \tau &= \frac{\tau'}{\sqrt{1-u^2/c^2}} \end{aligned}$$

But $$t = N\tau = N\tau'/(1-u^2/c^2)^{1/2} \text{ , therefore}$$

$$\nu_1 = \frac{N}{(1-u/c)} \cdot \frac{(1-u^2/c^2)^{1/2}}{N\tau'} \qquad \text{and } \nu_0 = 1/\tau'$$

but $1 - x^2 = (1-x)(1+x)$

$$\nu_1 = \nu_0 \left(\frac{1+\beta}{1-\beta}\right)^{1/2}$$

i.e. a **BLUESHIFT.**

For the man on the LEFT, $\beta' = -\beta$ therefore,

$$\nu_2 = \nu_0 \left(\frac{1-\beta}{1+\beta}\right)^{1/2}$$

i.e. a **REDSHIFT.**

19. Cosmological Implications of Doppler Shift

Look at the stars, Hubble saw a red-shift when looking at the emission lines. Edwin Hubble from 1920s–1940s studied spectral lines of a large number of stars in distant galaxies and using their known characteristic brightness he estimated their distance from Earth. He found most galaxies redshifted.

Hubble's Law

The recession velocity of the galaxies relative to our galaxy is proportional to their distance from earth.

$$u = HD$$

where D is the distance to a galaxy, u is the recession speed and H is the Hubble parameter or constant.

20. Big Bang, expansion of universe

$$H \approx 2.5 \times 10^{-18}\,\mathrm{s}^{-1}$$

If all stars and galaxies were moving away from each other, then an observer located on any one of them would report the same effect. In the Big Bang theory the universe started as a singularity and expanded rapidly. The theory predicts that the age of the universe $\sim H^{-1}$, or 13 billion years (see diagrams).

21. Examples

The Ca II spectral line measured from stars in the Virgo cluster is found to be at 398.3 nm. In the lab frame, this line is measured at 396.8 nm. What is the speed of recession of the cluster? By using cephid variable stars in the galaxy M100 (found in the Virgo cluster), astronomers have deduced that the galaxy is $48 \pm 6 \times 10^6$ light years away. Assuming Hubble's Law holds, calculate the value of the Hubble constant using this data.

The relativistic Doppler shift equation in term of wavelengths is:

$$\begin{aligned}
\lambda_1 &= \lambda_0 \left(\frac{1+u/c}{1-u/c}\right)^{1/2} \\
\left(\frac{\lambda_1}{\lambda_0}\right)^2 (1-\beta) &= (1+\beta) \\
\beta\left(1+\frac{\lambda_1^2}{\lambda_0^2}\right) &= \left(\frac{\lambda_1}{\lambda_0}\right)^2 - 1 \\
\beta &= \frac{\left(\frac{\lambda_1}{\lambda_0}\right)^2 - 1}{\left(\frac{\lambda_1}{\lambda_0}\right)^2 + 1}
\end{aligned}$$

Now from the data above, $\lambda_1/\lambda_0 = 398.3/396.8 = 1.0038$, therefore

$$\begin{aligned}
\beta &= \frac{(1.0038)^2 - 1}{(1.0038)^2 + 1} = 3.79 \times 10^{-3} \\
u &= 3.79 \times 10^{-3} c = 1.14 \times 10^6\,\mathrm{ms}^{-1}
\end{aligned}$$

There are $\pi \times 10^7$ seconds in a year, and so one light year is:

$$\begin{aligned}
1\mathrm{ly} &= (3.15 \times 10^7\mathrm{s})(3.0 \times 10^8\mathrm{ms}^{-1}) = 0.95 \times 10^{16}\,\mathrm{m} \\
D &= 48 \times 10^6 \times 0.95 \times 10^{16} = 4.56 \times 10^{23}\,\mathrm{m}
\end{aligned}$$

and so

$$H = \frac{u}{D} = \left(\frac{1.14 \times 10^6}{4.56 \times 10^{23}}\right) = 2.5 \times 10^{-18}\,\mathrm{s}^{-1}$$

These objects very far away, and the light reaching earth gives information about the universe millions or even billions of years ago. It also allows us to say that m_e and e have not changed in magnitude by more than 1:10^{12} /year.

Note that we can only use the Special Relativity relation for the Doppler shift to calculate the speed of recession of stars and galaxies for small redshifts ($\beta \ll 1$), the effects of General Relativity become important as the speed of recession increases. In the small β limit, the Doppler shift equation may be simplified to:

$$u = c\frac{\Delta\lambda}{\lambda}$$

22. Another way to derive the addition of velocities

From diagram, O_1 in S' at v_1, O_2 in S'' at v_2 relative to S' and v relative to S. Thus

$$\begin{aligned}
\nu_1 &= \nu_0 \left(\frac{1-\beta_1}{1+\beta_1}\right)^{1/2} \\
&= \nu_0 \gamma_1 (1-\beta_1) \\
\nu_2 &= \nu_1 \left(\frac{1-\beta_2}{1+\beta_2}\right)^{1/2} \\
\nu_2 &= \nu_0 \left(\frac{1-\beta_1}{1+\beta_1}\right)^{1/2} \left(\frac{1-\beta_2}{1+\beta_2}\right)^{1/2} \\
&= \nu_0 \left(\frac{1-\beta}{1+\beta}\right)^{1/2}
\end{aligned}$$

These must be the same, therefore

$$\begin{aligned}
\frac{1-\beta}{1+\beta} &= \left(\frac{1-\beta_1}{1+\beta_1}\right)\left(\frac{1-\beta_2}{1+\beta_2}\right) \\
\frac{1-\beta}{1+\beta} &= \frac{1-\beta_1-\beta_2+\beta_1\beta_2}{1+\beta_1+\beta_2+\beta_1\beta_2} \\
(1+\beta_1+\beta_2+\beta_1\beta_2)(1-\beta) &= (1-\beta_1-\beta_2+\beta_1\beta_2)(1+\beta) \\
-\beta_1-\beta_2+\beta+\beta(\beta_1\beta_2) &= \beta_1+\beta_2-\beta-\beta(\beta_1\beta_2)
\end{aligned}$$

So, collecting terms in β,

$$\begin{aligned}
2\beta(1+\beta_1\beta_2) &= 2\beta_1 + 2\beta_2 \\
\beta &= \frac{\beta_1+\beta_2}{1+\beta_1\beta_2} \\
v &= \frac{v_1+v_2}{1+\frac{v_1 v_2}{c^2}}
\end{aligned}$$