1st Year Relativity - Notes on Lectures 1 & 2

Lecture One

1. Maxwell and the speed of light.

   Maxwell's equations in free space produce wave equations of the form

   $$\nabla^2 \mathbf{E} = \frac{1}{c^2}\frac{\partial^2 \mathbf{E}}{\partial t^2} \qquad \text{and} \qquad \nabla^2 \mathbf{H} = \frac{1}{c^2}\frac{\partial^2 \mathbf{H}}{\partial t^2}$$

   This is a wave equation for $\mathbf{E}$ and $\mathbf{H}$ and has solutions of the form

   $$\begin{aligned} \mathbf{E} &= \mathbf{E}_0 \sin(\mathbf{k}.\mathbf{x} - \omega t) \\ \omega &= ck,\ k = 2\pi/\lambda \end{aligned}$$

   The phase is defined as

   $$\phi = \mathbf{k}.\mathbf{x} - \omega t$$

   and is an invariant.

   For a Galilean transformation from frame S to S' we have $\mathbf{x}' = \mathbf{x} - \mathbf{v}t$ and $t' = t$, thus for $\mathbf{x}$ along $\mathbf{k}$:

   $$\begin{aligned} \phi &= kx - \omega t \\ \left(\frac{\partial x}{\partial t}\right)_\phi &= \frac{\omega}{k} = c \\ \mathbf{k}'.\mathbf{x}' - \omega' t' &= \mathbf{k}.\mathbf{x} - \omega t \\ \therefore\ \mathbf{k}'.(\mathbf{x} - \mathbf{v}t) - \omega' t &= \mathbf{k}.\mathbf{x} - \omega t \\ \therefore\ \mathbf{k}' &= \mathbf{k},\ \omega' = \omega - \mathbf{v}.\mathbf{k}' = \omega - \mathbf{v}.\mathbf{k} = \omega(1 - \frac{\widehat{\mathbf{v}.\mathbf{k}}}{c}) \end{aligned}$$

   where $\hat{\mathbf{k}}$ is the unit vector perpendicular to the wavefront (c.f. Doppler shift). Now phase velocity in S' is

   $$c' = \frac{\omega'}{k'} = c - \mathbf{v}.\hat{\mathbf{k}} \neq c$$

   For propagation along the common axis,

   $$\sin(k'x' - \omega' t') = \sin(kx' - \omega(1 - \frac{v}{c})t')$$

   as $v \to c$ this equation predicts that an observer chasing the EM wave sees a field oscillating in space (sinusoidally) but not in time. This is not a solution of Maxwell's equations, therefore Maxwell's equations are not covariant under Galilean transformations.

2. One solution to this problem was the aether hypothesis, proposed in the latter half of the nineteenth century. Suppose Maxwell's equations were true in some "preferred" frame in which c was a constant and through the "medium" of which

EM waves propagated - the aether. Thus we have the possibility of **ABSOLUTE** motion.

Experiments were devised to measure the motion of the Earth relative to the aether e.g. Michelson and Morley (MM) in 1887 - achieved $v_{\rm rel} < 5$ kms^{-1}. Best result to date was achieved using Mössbauer spectroscopy, < 5 cms^{-1} (Isaak 1970).

The idea behind the MM experiment is to look for interference fringe shifts when a Michelson interferometer is rotated.

From diagram, t_1 is the time taken to reflect from M_1, which is moving at u relative to the aether so for the round trip

$$\begin{aligned} t_1 &= \frac{L}{c+u} + \frac{L}{c-u} \\ &= \frac{2Lc}{c^2-u^2} \quad \text{which for } u << c \text{ gives} \\ t_1 &\sim \frac{2L}{c}(1+\frac{u^2}{c^2}) \end{aligned}$$

Now let us consider the path perpendicular to the motion relative to the aether. From diagram t_2 is the time to go $2L$, so ,

$$\begin{aligned} \sqrt{L^2 + \left(\frac{ut_2}{2}\right)^2} &= \frac{ct_2}{2} \\ L^2 + \left(\frac{u^2}{4}\right) t_2^2 &= \left(\frac{c^2}{4}\right) t_2^2 \\ t_2^2 &= \frac{4L^2}{c^2-u^2} \\ t_2 &= \frac{2L/c}{\sqrt{1-u^2/c^2}} \end{aligned}$$

Now for small x,

$$\frac{1}{(1-x)^{\frac{1}{2}}} \approx 1 + \frac{1}{2}x$$

so for $x = u^2/c^2$, t_2 becomes $t_2 = \frac{2L}{c}(1+\frac{u^2}{2c^2})$ and the time difference for light travelling along the two paths becomes

$$\Delta t = t_1 - t_2 = \frac{2L}{c}\left[1+\frac{u^2}{c^2}\right] - \frac{2L}{c}\left[1+\frac{u^2}{2c^2}\right] = \frac{Lu^2}{c^3}$$

therefore $c\Delta t = Lu^2/c^2 =$ path length difference ΔL.

This will cause a shift in the fringes. In order to get rid of systematic errors the apparatus is turned through 90°, which leads to a doubling of the above path difference.

Therefore for MM experiment, $\Delta L = 2Lu^2/c^2$, therefore

$$\frac{\Delta L}{\lambda} = 2(L/\lambda)(u/c)^2$$

The apparatus can measure a shift of 0.04 fringes $\equiv (u/c)^2$ as small as 10^{-8}. For Earth velocity around Sun we would get 0.4 fringes - NO SHIFT FOUND!

3. **Einstein**

   Went to school aged 6, got the top grades - folklore wrong! 1888 went to Luitpold Gymnasium in Munich. Earned highest or next highest marks in maths and Latin for whole time that he was in school. He didn't like school, especially teachers! One report said "he had a natural antipathy for ... gymnastics and sports ... he easily became dizzy and tired".

   He never read "light" literature - always science or the writings of Kant and music. Aged 12 he read through a book on Euclidean geometry. Went to university in Zurich in 1896, qualified as a specialist teacher in maths and physics (top grades) in 1900.

   In 1900 he finished his first paper on inter-molecular forces, gained Swiss citizenship in 1901. He tried for a university position but was turned down. Went to Bern in 1902 to the Federal Patent Office. Published papers whilst there in 1903 and 1904 on the foundations of statistical mechanics. 1905 was a great year for him - Nobel prize for paper on light quanta, in April he got his Ph.D. from Zurich on K.T. and the size of molecules, June - special relativity paper. 1906 - specific heats, 1907 - general relativity.

4. **History of Relativity**

   In 1851 Fizeau performed experiments on light in moving liquids. November 1887 MM experiment -ve result a great disappointment to Kelvin, Rayleigh and Lorentz.

   Fitzgerald: "... the length of material bodies changes, according as they are moving through the aether ..." **N.B.** Still aether, not the **relative** change in L_0.

   Lorentz said: Aether hypothesis is OK if, in MM experiment, on 90 deg rotation of the apparatus, L becomes L' where:

   $$L' = L(1 - v^2/2c^2)$$

   This is the Lorentz - Fitzgerald contraction, correct to second order in v/c. Lorentz derived his famous force equation in 1895:

   $$\mathbf{F} = q(\mathbf{E} + \mathbf{v} \wedge \mathbf{B})$$

   Einstein knew all this, but not what follows:

   1899 Lorentz writes LT to a scale factor ε

   1904 Gets it right!

   Larmor in 1900 produced LT independently and predicted L-FG contraction.

   Poincaré - 1898 article "we have no direct intuition about the equality of two time intervals". Didn't make leap to special relativity. In 1905 he got the transformation of velocities right.

   Einstein 1905 (June) finally stated the two principles of relativity.

5. Einstein's two postulates of the Special Theory of Relativity.

   (1) The laws of physics are the same in all inertial frames

(2) The speed of light in empty space is the same in all inertial frames and is independent of the motion of its source.

Lecture Two

6. Derivation of the Lorentz Transformation - N.B. Derivation **NOT** required in Prelims - only result!

   Linear transformation. Two frames S and S', S' moving at speed v along x (neglect y and z for now) w.r.t. S. What assumptions am I making?

   (a) The speed of light is the same in all inertial frames of reference.

   (b) The transformation depends only on the relative velocity of the two frames of reference and is symmetric w.r.t. them.

   (c) The transformation is a linear function, involving only first powers of the coordinates x, x and t, t' in S and S'.

   Let:

$$L(v) = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \qquad (1)$$

$$\begin{pmatrix} x' \\ t' \end{pmatrix} = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} x \\ t \end{pmatrix} \qquad (2)$$

$$x' = Ax + Bt \qquad (3)$$

$$t' = Cx + Dt \qquad (4)$$

   Coefficients B and C interrelate x and t and therefore must change sign with v. Thus

$$L(-v) = \begin{pmatrix} +A & -B \\ -C & +D \end{pmatrix}$$

   By condition (b)

$$L(v)L(-v) = 1$$

   therefore

$$\begin{pmatrix} A^2 - BC & -AB + BD \\ CA - DC & -CB + D^2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad (5)$$

$$A = D \qquad (6)$$

$$A^2 - BC = 1 \qquad (7)$$

   The origin of one frame of reference moves with velocity v in the other, so that if $x' = 0$, $x = vt$. Therefore

$$Avt + Bt = 0 \qquad (8)$$

$$B = -Av \qquad (9)$$

Now c is the same in both systems, so

$$\frac{x}{t} = c = \frac{x'}{t'} \tag{10}$$

$$c = \frac{Ax+Bt}{Cx+Dt} \quad \text{(eqn 3/eqn 4)} \tag{11}$$

$$c = \frac{Ac+B}{Cc+D} \quad (x=ct) \tag{12}$$

but $A = D$ and $B = -Av$, therefore

$$Ac - Av = Cc^2 + Ac \tag{13}$$

$$C = \frac{-Av}{c^2} \tag{14}$$

but $A^2 - BC = 1$ therefore

$$A^2 - \frac{A^2v^2}{c^2} = 1 \tag{15}$$

$$A^2 = \frac{1}{1 - v^2/c^2} \tag{16}$$

therefore

$$A = D = \frac{1}{\sqrt{1-v^2/c^2}} = \gamma$$

and

$$C = \frac{-v/c^2}{\sqrt{1-v^2/c^2}} = \frac{-\gamma v}{c^2}$$

let $\beta = v/c$, then

$$L(v) = \begin{pmatrix} \gamma & -\gamma v \\ -\gamma v/c^2 & \gamma \end{pmatrix} = \begin{pmatrix} \gamma & -c\beta\gamma \\ -\gamma\beta/c & \gamma \end{pmatrix}$$

Now replace t by ict and get

$$\mathcal{L}(v) = \begin{pmatrix} \gamma & i\beta\gamma \\ -i\beta\gamma & \gamma \end{pmatrix} \tag{17}$$

$$\begin{pmatrix} x' \\ ict' \end{pmatrix} = \mathcal{L}(v)\begin{pmatrix} x \\ ict \end{pmatrix} \tag{18}$$

In this proof we already have $L(v)L(-v) = 1$ so the transforms are symmetric by definition! Thus the inverse transform is obtained by swapping primes and writing $-v$ for v. Exercise - check this!

7. What about y and z? Let

$$y = \alpha y' \tag{19}$$

$$z = \beta z' \tag{20}$$

Then also we must have, by the relativity principle,

$$y' = \alpha y \tag{21}$$
$$z' = \beta z \tag{22}$$

Then $yy' = \alpha\alpha y' y$, so

$$\alpha^2 = 1 \tag{23}$$
$$\alpha = \pm 1 \tag{24}$$
$$\beta = \pm 1 \tag{25}$$

Thus $y = y'$ and $z = z'$

8. What about the classical limit? - these transformations must still obey Newton's Laws in this situation. Take the limit $v \ll c$,

$$\gamma = (1 - v^2/c^2)^{-\frac{1}{2}} \tag{26}$$
$$\approx 1 + \frac{v^2}{2c^2} \tag{27}$$

therefore to first order in v/c, $\gamma \to 1$ and

$$x' = x - vt \tag{28}$$
$$t' = t - \frac{vx}{c^2} \leftarrow 0 \tag{29}$$
$$t' = t \tag{30}$$

Thus we get the Galilean transformations back.

9. Now lets see why I chose to use *ict* earlier. Imagine in each frame a spherical wave spreading from the origin, they have the form:

$$x^2 + y^2 + z^2 - c^2t^2 = 0 \tag{31}$$
$$x'^2 + y'^2 + z'^2 - c^2t'^2 = 0 \tag{32}$$

therefore for events on an expanding wavefront

$$x^2 - c^2t^2 = -(y^2 + z^2) \quad |x| \leq ct \tag{33}$$
$$x'^2 - c^2t'^2 = -(y'^2 + z'^2) \quad |x'| \leq ct' \tag{34}$$
$$y = y' \tag{35}$$
$$z = z' \tag{36}$$
$$x^2 - c^2t^2 = x'^2 - c^2t'^2 \tag{37}$$

This is an example of invariance in special relativity, the square of the 4-vector is invariant under Lorentz transformation.

The next section is a mathematical treatment intended to show some mathematical beauty and consistency in special relativity. It is intended strictly for those with a mathematical bent and is in no way on the syllabus. You may safely ignore all of the following section — you have been warned!

10. Now lets look at the LT again but from a different point of view. If we have a universal speed, why not talk only in fractions of this, i.e. in terms of v/c? Lets talk about space and time as one co-ordinate system so that we have:

$$x_1 = x,\ x_2 = y,\ x_3 = z,\ x_4 = ict$$

Then a rotation in the (x_1, x_4) plane through an angle θ transforms $(x_1 \ldots x_4) \rightarrow (x_1' \ldots x_4')$. What is a rotation matrix (ignoring y and z)?

$$\begin{pmatrix} x_1' \\ x_4' \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} x_1 \\ x_4 \end{pmatrix}$$

which when expanded gives

$$x' = x\cos\theta + ict\sin\theta \quad (38)$$
$$ict' = -x\sin\theta + ict\cos\theta \quad (39)$$

NOT very useful - look at all the $\sqrt{-1}$! But:

$$\cos i\phi = \cosh\phi \quad (40)$$
$$\sin i\phi = i\sinh\phi \quad (41)$$

Therefore replace θ by $i\phi$. Thus

$$x' = x\cosh\phi - ct\sinh\phi \quad (42)$$
$$t' = -\frac{x}{c}\sinh\phi + t\cosh\phi \quad (43)$$

or rewriting in x, ict form,

$$\begin{pmatrix} x' \\ ict' \end{pmatrix} = \begin{pmatrix} \cosh\phi & i\sinh\phi \\ -i\sinh\phi & \cosh\phi \end{pmatrix} \begin{pmatrix} x \\ ict \end{pmatrix}$$

where now $\tanh\phi = v/c = \beta$. Then

$$\gamma = \cosh\phi = (1 - \tanh^2\phi)^{-\frac{1}{2}} \quad (44)$$
$$\sinh\phi = \gamma\tanh\phi = \gamma\beta \quad (45)$$
$$\begin{pmatrix} x' \\ ict' \end{pmatrix} = \begin{pmatrix} \gamma & i\beta\gamma \\ -i\beta\gamma & \gamma \end{pmatrix} \begin{pmatrix} x \\ ict \end{pmatrix} \quad (46)$$

The pseudo angle ϕ represents the rapidity - you can add ϕs but you can't add vs, i.e. rotations add O.K. Thus to transform two velocities v_1 and v_2 to v use $\phi_1 + \phi_2 = \phi$. Using a trig. relation we have:

$$\tanh(\phi_1 + \phi_2) = \frac{\tanh\phi_1 + \tanh\phi_2}{1 + \tanh\phi_1\tanh\phi_2} \quad (47)$$
$$\frac{v_1}{c} = \tanh\phi_1 \quad (48)$$
$$\frac{v}{c} = \tanh\phi = \tanh(\phi_1 + \phi_2) \quad (49)$$
$$\frac{v}{c} = \frac{v_1/c + v_2/c}{1 + v_1 v_2/c^2} \quad (50)$$
$$v = \frac{v_1 + v_2}{1 + v_1 v_2/c^2} \quad (51)$$

which is the relativistic transformation of velocities - check in a text book. Let's look at ϕ for a moment,

$$\phi = \tanh^{-1}(v/c)$$

ϕ is continuous, differentiable, strictly monotonic, is equal to 0 when $v = 0$ and tends to ∞ as $v \to c$ and $-\infty$ as $v \to -c$. We can therefore convert from speeds to rapidities - which are simply additive.

We'll come back to the addition of velocities later in the lecture course.