The wave equation - solution by separation of variables

$$\frac{\partial^2 y}{\partial x^2} = \frac{1}{\mathrm{v}^2}\frac{\partial^2 y}{\partial t^2}$$

Try $y(x,t) = X(x)T(t)$

$$T(t)\frac{d^2X(x)}{dx^2} = \frac{1}{\mathrm{v}^2}X(x)\frac{d^2T(t)}{dt^2} \quad \Rightarrow \quad \frac{\ddot{X}}{X} = \frac{1}{\mathrm{v}^2}\frac{\ddot{T}}{T}$$

Hence $\frac{\ddot{X}}{X} = \frac{1}{v^2}\frac{\ddot{T}}{T} = C_S = -k^2$, C_S "Separation Constant" (taken -ve here)

A,B,C,D constants

$$\ddot{X} + k^2 X = 0 \qquad\qquad X = A\sin kx + B\cos kx$$
$$\Rightarrow$$
$$\ddot{T} + k^2\mathrm{v}^2 T = 0 \qquad\qquad T = C\sin k\mathrm{v}t + D\cos k\mathrm{v}t$$

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k complex possible too

Hence $\frac{\ddot{X}}{X} = \frac{1}{\mathrm{v}^2}\frac{\ddot{T}}{T} = C_S = +k^2$, C_S "Separation Constant" (taken +ve here)

A,B,C,D constants

$$\ddot{X} + k^2 X = 0 \qquad X = Ae^{kx} + Be^{-kx}$$

$$\ddot{T} + k^2 v^2 T = 0 \quad \Rightarrow \quad T = Ce^{kvt} + De^{-kvt}$$

For vibrating string physically relevant case is -k^2

$$y(x,t) = X(x)T(t) = (A\sin kx + B\cos kx)(C\sin k\mathrm{v}t + D\cos k\mathrm{v}t)$$
$$= A'(\sin kx + B'\cos kx)(\sin k\mathrm{v}t + D'\cos k\mathrm{v}t)$$

Relation to d'Alembert's solution?

$$y(x,t) = \alpha\cos(kx+\delta)\cos(k\mathrm{v}t+\rho), \qquad (\alpha,\rho,\delta \text{ constants})$$

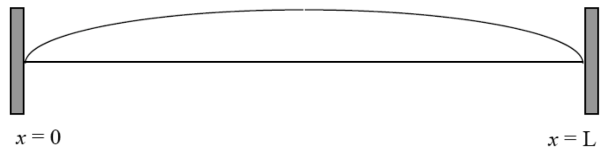
$$= \frac{\alpha}{2}\left[\cos(k(x+\mathrm{vt})+\delta+\rho) + \cos(k(x-\mathrm{vt})+\delta-\rho)\right]$$

$$(= f(x+\mathrm{v}t) + g(x-\mathrm{v}t))$$

Example 1

String with fixed ends initially displaced and at rest

$$\frac{\partial^2 y}{\partial x^2} = \frac{1}{v^2}\frac{\partial^2 y}{\partial t^2}$$



Initial conditions : $y(x,t=0) = h(x)$, $\frac{\partial y}{\partial t}(x,t=0) = 0$

End points: $y(0,t)=y(L,t)=0$, for all t

$$y(x,t) = (A\sin kx + B\cos kx)(C\sin kvt + D\cos kvt)$$

$$y(0,t) = 0 \quad \Rightarrow \quad B=0$$

$$\frac{\partial y}{\partial t}(x,t=0) = 0 \quad \Rightarrow \quad C = 0$$

$$y(L,t) = 0 \quad \Rightarrow \quad kL = n\pi, \; n \in Z$$

Eigenvalue equation

Principle of superposition

$$y(x,t) = \sum_{n=1}^{\infty} A_n \sin\frac{n\pi x}{L}\cos\frac{n\pi vt}{L}$$

A_n constants

$$y(x,t) = \sum_{n=1}^{\infty} A_n \sin\frac{n\pi x}{L}\cos\frac{n\pi v t}{L}$$

A_n fixed by initial conditions

$$y(x,0) = \sum_{n=1}^{\infty} A_n \sin\frac{n\pi x}{L} = h(x)$$

$eg(a)$ $h(x) = B\sin(m\pi x / L)$ $\quad A_m = B, \quad A_{n\neq m} = 0$

$$y(x,t) = B\sin\frac{m\pi x}{L}\cos\frac{m\pi v t}{L}$$

Standing wave - single normal mode excited

$$y(x,t)=\sum_{n=1}^{\infty} A_n \sin\frac{n\pi x}{L}\cos\frac{n\pi \text{v}t}{L}$$

A_n fixed by initial conditions

$$y(x,0)=\sum_{n=1}^{\infty} A_n \sin\frac{n\pi x}{L}=h(x)$$

eg(b) $h(x)=\sin\frac{\pi x}{L}+\frac{1}{2}\sin\frac{2\pi x}{L}$ $\quad A_1=1,\quad A_2=\frac{1}{2},\quad A_{n\neq 1,2}=0$

$$y(x,t)=\sin\frac{\pi x}{L}\cos\frac{\pi v t}{L}+\frac{1}{2}\sin\frac{2\pi x}{L}\cos\frac{2\pi v t}{L}$$

N.B. For non-NM initial displacement, subsequent motion is NOT EQUAL to initial displacement $\times$ varying amplitude.

Shorter wavelengths oscillate faster (constant speed) -Hence shape of wave varies during oscillation.

In[1]:=
```
Animate[Plot[Sin[π x] Cos[π t] + Sin[2 π x] Cos[2 π t], {x, 0, 2}],
 {t, 0, 2}, AnimationRunning → False]
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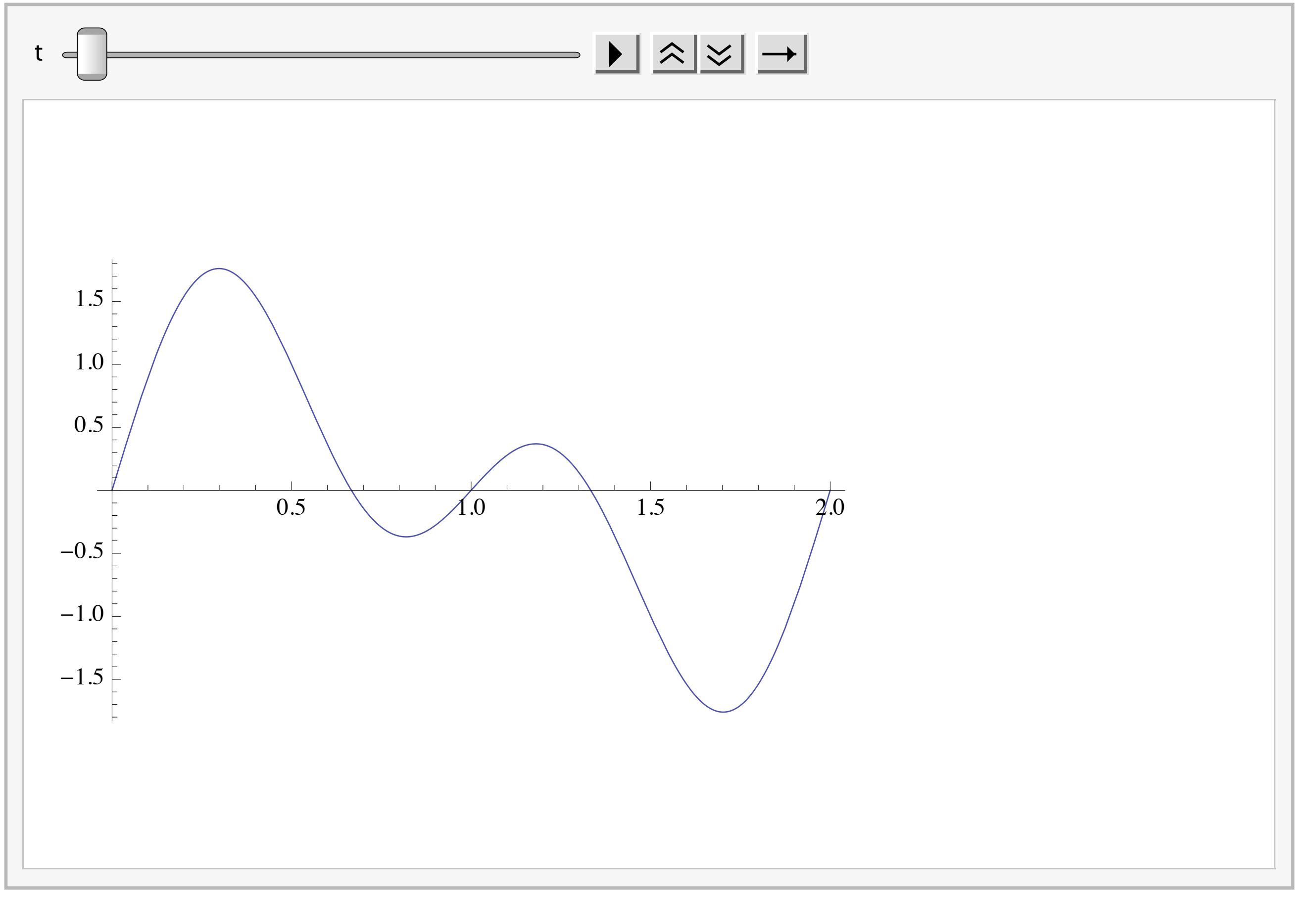
Out[1]=



$$y(x,t)=\sum_{n=1}^{\infty} A_n \sin\frac{n\pi x}{L}\cos\frac{n\pi vt}{L}$$

"Fourier series" (2nd year)

A_n fixed by initial conditions

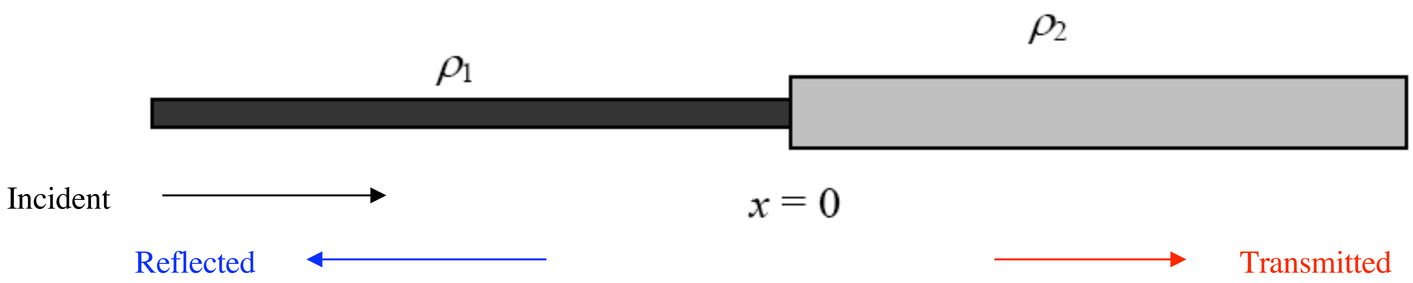
$$y(x,0)=\sum_{n=1}^{\infty} A_n \sin\frac{n\pi x}{L} = h(x)$$

General solution

$$A_n = \frac{2}{L}\int_0^L h(x)\sin\frac{n\pi x}{L}\,dx$$

Follows from orthogonality of sines

$$\frac{2}{L}\int_0^L \sin\frac{m\pi x}{L}\sin\frac{n\pi x}{L}\,dx = \begin{cases} 0, \text{ if } m \neq n \\ 1, \text{ if m=n} \end{cases}$$

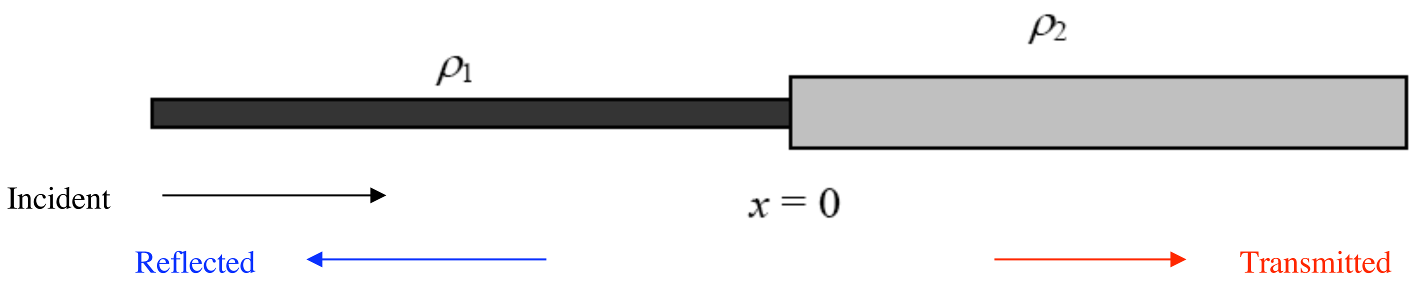


$$\psi_{incident} = A\sin(\omega t - k_1 x)$$ $\mathrm{Im}\{Ae^{i\{k_1 x - wt\}}\}$ (A, A', A'' real here)

$$\psi_{reflected} = A'\sin(\omega t + k_1 x)$$ $\mathrm{Im}\{A'e^{-i\{k_1 x + wt\}}\}$

$$\psi_{transmitted} = A''\sin(\omega t - k_2 x)$$ $\mathrm{Im}\{A''e^{i\{k_2 x - wt\}}\}$

- All waves have same ω - necessary to satisfy boundary condition at x=0
- Right moving waves $-k_1 x$
- Left moving waves $+k_2 x$



$$\psi_{incident} = A\sin\left(\omega t - k_1 x\right)$$

$$\psi_{reflected} = A'\sin\left(\omega t + k_1 x\right)$$

$$\psi_{transmitted} = A''\sin\left(\omega t - k_2 x\right)$$

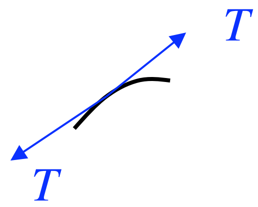
Boundary conditions

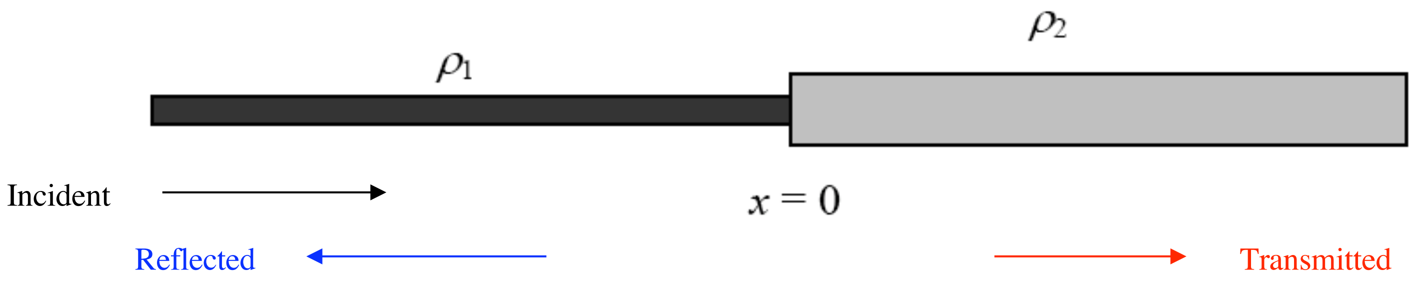
$$y(x=0-,t) = y(x=0+,t)$$

String continuous

$$\frac{\partial y}{\partial x}(x=0-,t) = \frac{\partial y}{\partial x}(x=0+,t)$$

Forces continuous





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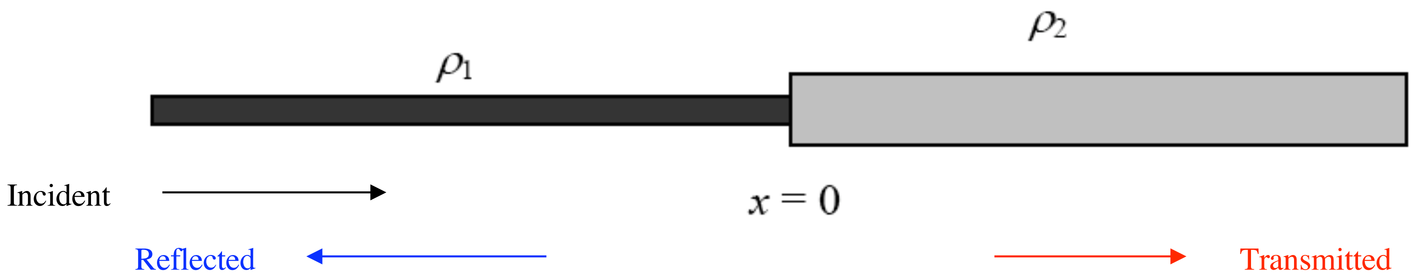
$$\psi_{transmitted} = A''\sin\left(\omega t - k_2 x\right)$$

Boundary conditions

$$y(x = 0-, t) = y(x = 0+, t) \qquad \psi_{incident}(x = 0, t) + \psi_{reflected}(0, t) = \psi_{transmitted}(0, t)$$

$$\frac{\partial y}{\partial x}(x = 0-, t) = \frac{\partial y}{\partial x}(x = 0+, t) \qquad \frac{\partial \psi_{incident}(x = 0, t)}{\partial x} + \frac{\partial \psi_{reflected}(0, t)}{\partial x} = \frac{\partial \psi_{transmitted}(0, t)}{\partial x}$$



$$\psi_{incident} = A\sin(\omega t - k_1 x)$$

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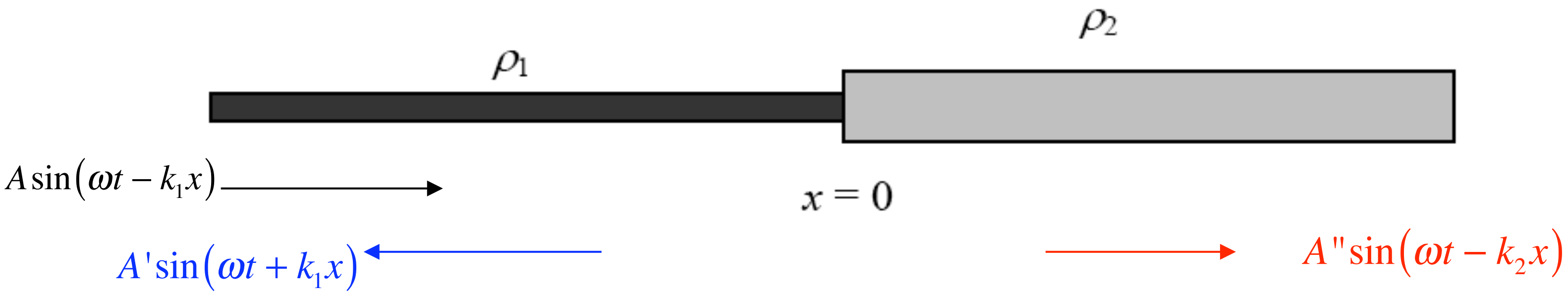
Boundary conditions

$$y(x=0-,t) = y(x=0+,t) \qquad \psi_{incident}(x=0,t) + \psi_{reflected}(0,t) = \psi_{transmitted}(0,t)$$

$$\frac{\partial y}{\partial x}(x=0-,t) = \frac{\partial y}{\partial x}(x=0+,t) \qquad \frac{\partial \psi_{incident}(x=0,t)}{\partial x} + \frac{\partial \psi_{reflected}(0,t)}{\partial x} = \frac{\partial \psi_{transmitted}(0,t)}{\partial x}$$

$$A + A' = A''$$

$$k_1(A - A') = k_2 A''$$



$$A + A' = A''$$

$$k_1(A - A') = k_2 A''$$

$$r = \frac{A'}{A} = \frac{k_1 - k_2}{k_1 + k_2}$$ reflection amplitude

$$t = \frac{A''}{A} = \frac{2k_1}{k_1 + k_2}$$ transmission amplitude

Special cases:

1) $k_1 = k_2 \Rightarrow \quad A' = 0, \; t = \frac{A''}{A} = 1$ No reflection

2) $k_1 < k_2 \quad \Rightarrow A'$ is negative Reflected wave $= -|A'|\sin(\omega t + k_1 x) = |A'|\sin(\omega t + k_1 x + \pi)$

i.e. PHASE CHANGE at rare-dense boundary $(k_1 < k_2 \Rightarrow \rho_1 < \rho_2)$ $[\, v = \omega / k = \sqrt{T/\rho}\,]$

3) $k_1 > k_2 \quad \Rightarrow A'$ is positive

4) $\rho_2 \to \infty \Rightarrow \quad k_2 \to \infty \qquad r = \frac{A'}{A} \to -1 \qquad T \to 0$ No wave in very heavy string

Energy flux at boundaries

Power flux $P = \frac{1}{2} T \omega k A^2$ (Energy flow per unit time)

Incident power flux $P_I = \frac{1}{2} T \omega k_1 A^2$

Reflected power flux $P_R = \frac{1}{2} T \omega k_1 A'^2$

Transmitted power flux $P_T = \frac{1}{2} T \omega k_2 A''^2$

$$R_P = \frac{P_R}{P_I} = \frac{A'^2}{A^2} = r^2 = \left(\frac{k_1 - k_2}{k_1 + k_2} \right)^2$$ Coefficient of reflection

$$T_P = \frac{P_T}{P_I} = \frac{k_2 A''^2}{k_1 A^2} = \frac{k_2}{k_1} t^2 = \frac{k_2}{k_1} \left(\frac{2k_1}{k_1 + k_2} \right)^2 = \frac{4k_1 k_2}{\left(k_1 + k_2\right)^2}$$ Coefficient of transmission

$$R_P + T_P = \left(\frac{k_1 - k_2}{k_1 + k_2} \right)^2 + \frac{4k_1 k_2}{\left(k_1 + k_2\right)^2} = \frac{k_1^2 + 2k_1 k_2 + k_2^2}{\left(k_1 + k_2\right)^2} = 1$$ Conservation of energy