Information transmission

Plane wave does not convey information - must be modulated

e.g.

$$y = A\sin(kx - \omega t), \quad |kx - \omega t| \le \omega T / 2$$

$$= 0, \quad |kx - \omega t| > \omega T / 2$$

To construct this modulated wave need a wave packet made up of an infinite number of frequencies - Fourier series (2nd year topic)

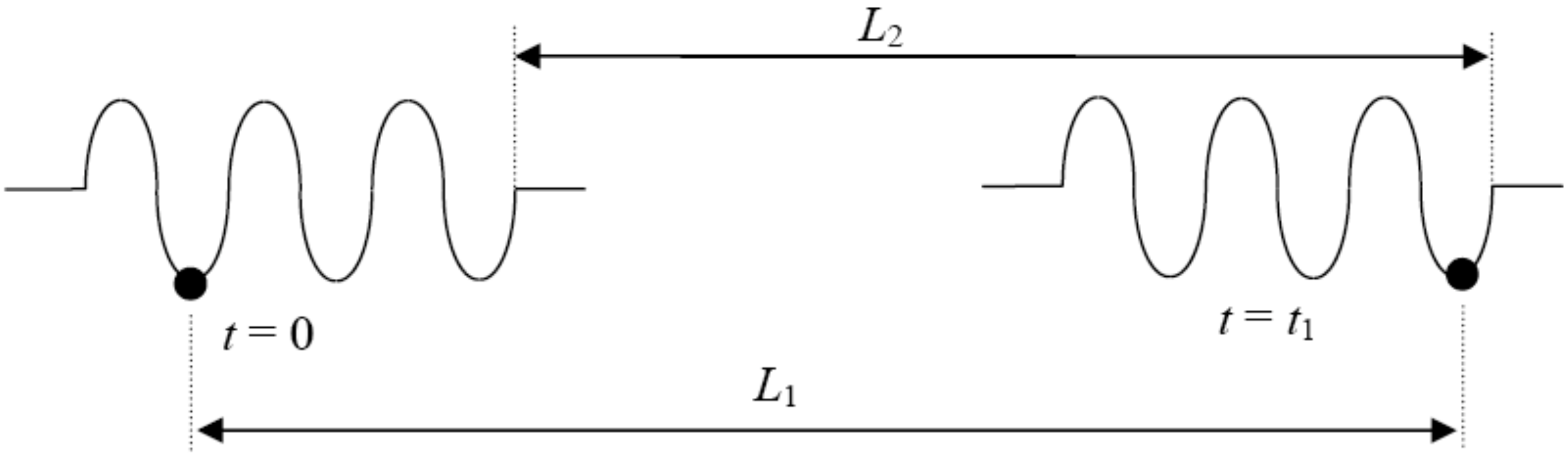
$$y(x,t) = \sum_{n=1}^{N} D_n \cos(k_n x - \omega_n t)$$

Rate if information transfer : velocity of envelope … "GROUP VELOCITY"

c.f. speed of individual waves of definite frequency … "PHASE VELOCITY" $\left(v = \frac{\omega_n}{k_n}\right)$

Group and phase velocity

$$y(x,t)=\sum_{n=1}^{N} D_n \cos\left(k_n x-\omega_n t\right)$$



Group velocity, g....... $g=\frac{L_2}{t_1}$

Phase velocity, v..... *e.g.* $v=\frac{L_1}{t_1}$ (Since many frequencies need not be the same as the group velocity)

$$y(x,t) = \sum_{n=1}^{N} D_n \cos\left(k_n x - \omega_n t\right)$$

Non-dispersive medium

$$\mathrm{v}_{\mathrm{n}} = \mathrm{v}$$

$$y(x,t) = \sum_{n=1}^{N} D_n \cos k_n \left(x - \mathrm{v}t\right) \equiv y(x - \mathrm{v}t) \quad \Rightarrow \quad \omega_n = k_n \mathrm{v} \quad \Rightarrow \quad \frac{\delta\omega}{\delta k} \equiv \frac{\omega}{k} = \mathrm{v}$$

$$i.e.\ g = \mathrm{v}$$

e.g Light through glass prism
Different colours ➔ different angles, because the refractive index μ (= c/v) depends on colour (ω), i.e.v depends on ω.

Dispersive medium

$$\mathrm{v}_{\mathrm{n}} \neq \mathrm{v}$$

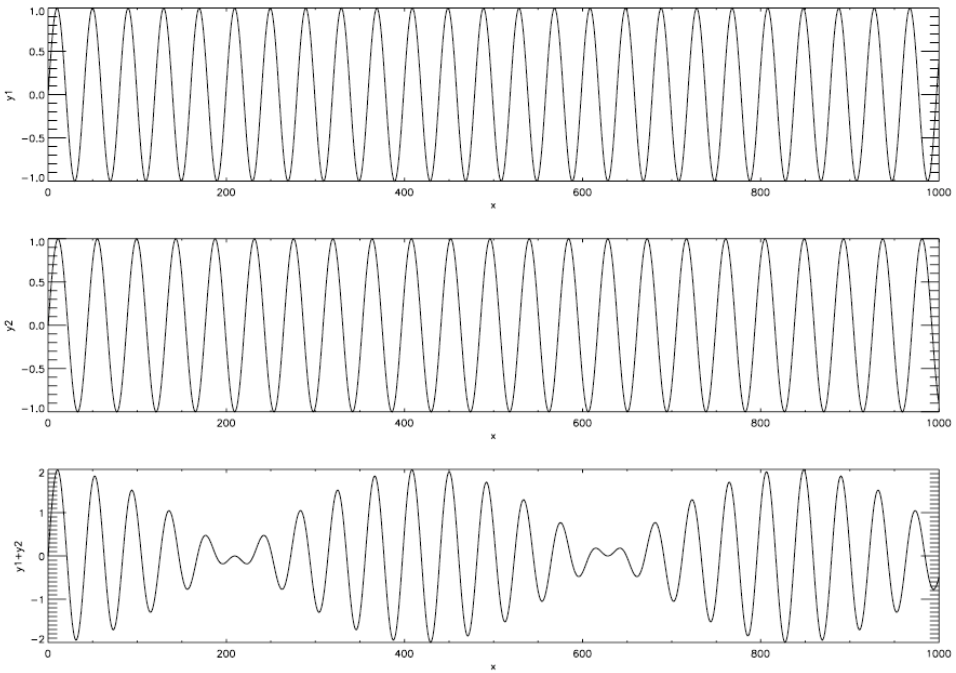
$$i.e.\ g \neq \mathrm{v}$$

- Simple example : superposition of two waves

$$y_1 = A\sin\left[\left(k+\delta k\right)x-\left(\omega+\delta\omega\right)t\right]$$

$$y_2 = A\sin\left[\left(k-\delta k\right)x-\left(\omega-\delta\omega\right)t\right]$$

$$\delta k << k, \quad \delta\omega << \omega$$

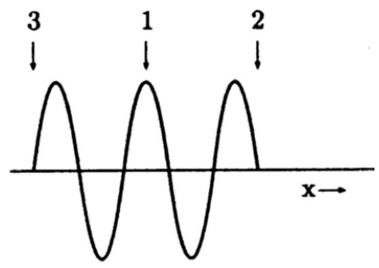


$$y = y_1 + y_2 = 2A\cos\left(\delta k.x-\delta\omega.t\right)\sin\left(kx-\omega t\right)$$

Phase velocity $\mathrm{v} \simeq \dfrac{\omega}{\mathrm{k}}$

Group velocity $g = \dfrac{\delta\omega}{\delta k}$ (velocity of envelope)

To construct a finite pulse need a superposition of an infinite number of waves of different frequencies



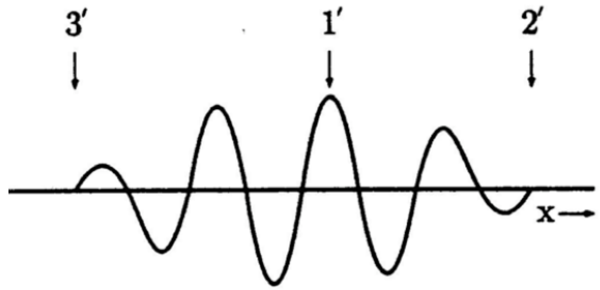
If distribution is peaked around ω, k

Phase velocity $v \simeq \dfrac{\omega}{k}$

Group velocity $g = \dfrac{d\omega}{dk}$ (velocity of envelope)

In a non-dispersive medium pulse maintains its shape $\dfrac{d\omega}{dk} = \dfrac{\omega}{k} = \text{constant}$

In a dispersive medium pulse spreads out
… but mean position moves with group velocity



$N.B. \quad g \leq c$

There are many equivalent expressions for the group velocity:

$$g = \frac{d\omega}{dk}$$

$$\omega = \mathrm{v}k \qquad g = \mathrm{v} + k\frac{d\mathrm{v}}{dk}$$

$$k = 2\pi / \lambda \qquad g = \mathrm{v} - \lambda\frac{d\mathrm{v}}{d\lambda}$$

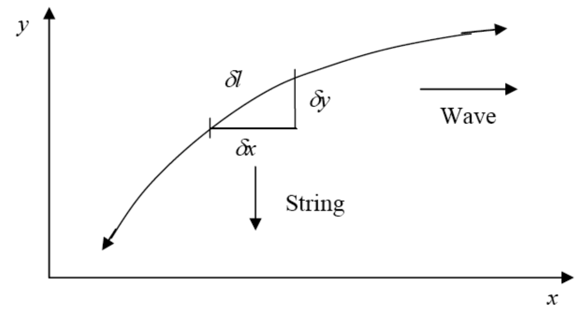
$$\mathrm{v} = c / \mu \qquad g = \frac{c}{\mu}\left(1 + \frac{\lambda}{\mu}\frac{d\mu}{d\lambda}\right)$$

In terms of wavelength λ' in vacuum

$$g = \mathrm{v}\left(1 - \frac{1}{1 + \frac{\mathrm{v}}{\lambda'}\frac{d\lambda'}{d\mathrm{v}}}\right) \qquad \lambda' = \lambda\frac{c}{\mathrm{v}} \qquad f = \frac{c}{\lambda'} = \frac{\mathrm{v}}{\lambda}$$

Energy of vibrating string

$$y = A\sin\left(kx - \omega t\right)$$



Kinetic Energy

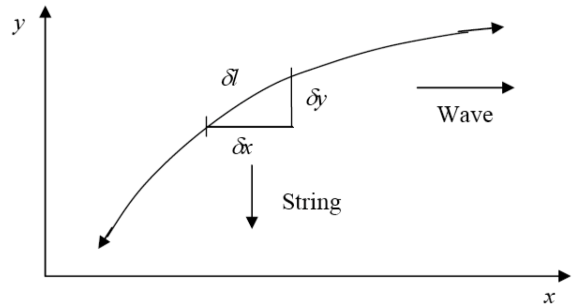
$$KE \text{ of section} = \frac{1}{2}\rho\delta x\left(\frac{\partial y}{\partial t}\right)^2 = \frac{1}{2}\rho A^2\omega^2\cos^2\left(kx - \omega t\right)\delta x$$

$$KE = \frac{1}{2}\rho A^2\omega^2\int_x^{x+l\lambda}\cos^2\left(kx' - \omega t\right)dx' = \frac{1}{2}\rho A^2\omega^2 \times \frac{l\lambda}{2}$$

(integer l wavelengths)

Energy of vibrating string

$$y = A\sin(kx - \omega t)$$



$$\frac{\partial^2 y}{\partial x^2} = \frac{\rho}{T}\frac{\partial^2 y}{\partial t^2} \equiv \frac{1}{c^2}\frac{\partial^2 y}{\partial t^2}$$

Kinetic Energy

$$KE \text{ of section} = \frac{1}{2}\rho\delta x\left(\frac{\partial y}{\partial t}\right)^2 = \frac{1}{2}\rho A^2\omega^2\cos^2(kx - \omega t)\delta x$$

$$KE = \frac{1}{2}\rho A^2\omega^2\int_x^{x+l\lambda}\cos^2(kx' - \omega t)dx' = \frac{1}{2}\rho A^2\omega^2 \times \frac{l\lambda}{2}$$

$$KE / l\lambda = \frac{1}{4}\rho A^2\omega^2$$

Potential Energy

$$PE \text{ in stretched string element} = T(\delta l - \delta x) = T\delta x\left(\sqrt{1+\left(\frac{\delta y}{\delta x}\right)^2} - 1\right) \approx \frac{1}{2}TA^2k^2\cos^2(kx - \omega t)\delta x$$

$$PE = \frac{1}{2}TA^2k^2\int_x^{x+l\lambda}\cos^2(kx' - \omega t)dx'$$

$$PE / l\lambda = \frac{1}{4k}TA^2k^2$$

$\mathrm{v} = \omega / k = \sqrt{T/\rho} \Rightarrow Tk^2 = \rho\omega^2 \Rightarrow$ PE=KE! (Example of virial theorem)

Energy flow

Total energy per wavelength, E/λ= KE+PE=$\frac{1}{2}\rho A^2\omega^2$

Distance travelled = vt

$$\text{Energy flow/unit time} = \left(\frac{1}{2}\rho A^2\omega^2\right)\text{v}t/t$$

$$= \frac{1}{2}\rho\omega^2 A^2 \text{v}$$

$$= \frac{1}{2}\rho\omega^3 A^2 / k \qquad \text{v} = \frac{\omega}{k}$$

$$= \frac{1}{2}Tk^2 A^2 \text{v} \qquad Tk^2 = \rho\omega^2$$

$$= \frac{1}{2}T\omega k A^2$$