D'Alembert's solution

$$\frac{\partial^2 y}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 y}{\partial t^2} \qquad \Rightarrow \qquad y(x,t) = f(x-ct) + g(x+ct)$$

f and g are determined by the initial conditions:

Suppose at time $t = 0$, the wave has an initial displacement $U(x)$ and an initial velocity $V(x)$

$$y(x,0) = f(x) + g(x) = U(x)$$

$$\frac{\partial y(x,0)}{\partial t} = -cf'(x) + cg'(x) = V(x) \quad \Rightarrow \quad f(x) - g(x) = -\frac{1}{c}\int_b^x V(x')\,dx'$$

$$f(x) = \frac{1}{2}U(x) - \frac{1}{2c}\int_b^x V(x')\,dx'$$

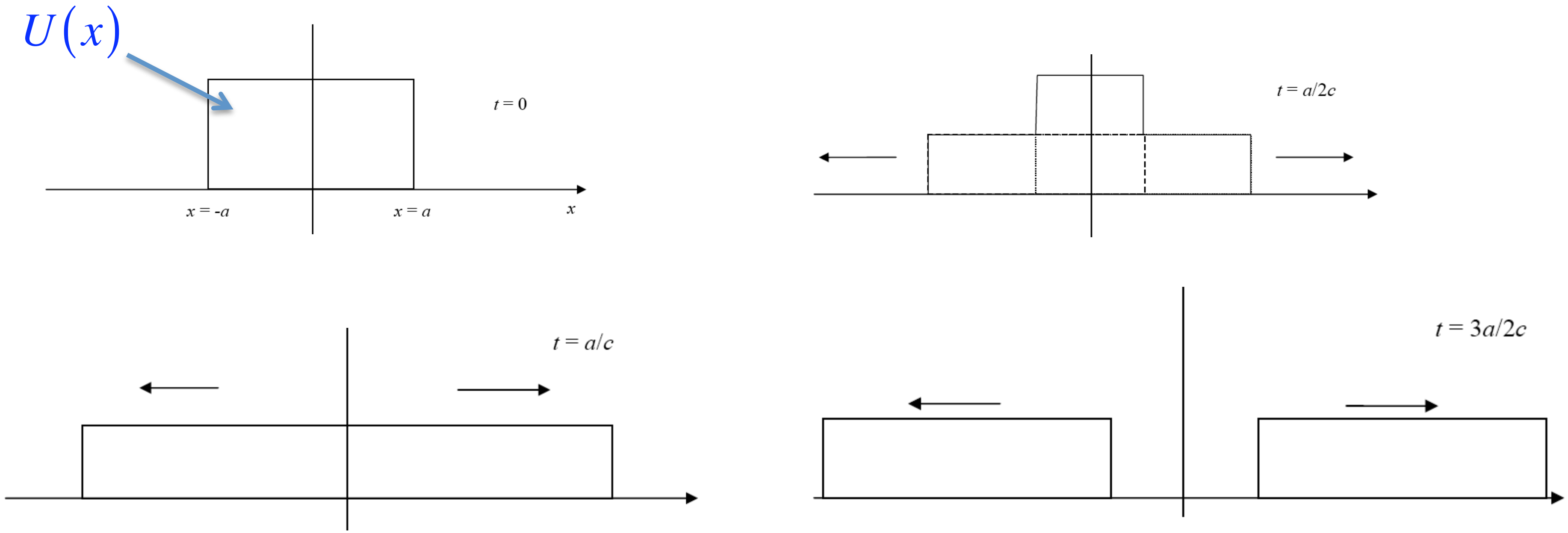
$$g(x) = \frac{1}{2}U(x) + \frac{1}{2c}\int_b^x V(x')\,dx'$$

$$y(x,t) = \frac{1}{2}\left[U(x-ct) + U(x+ct)\right] + \frac{1}{2c}\left[\int_b^{x+ct} V(x)\,dx - \int_b^{x-ct} V(x)\,dx\right] = \frac{1}{2}\left[U(x-ct) + U(x+ct)\right] + \frac{1}{2c}\int_{x-ct}^{x+ct} V(x)\,dx$$

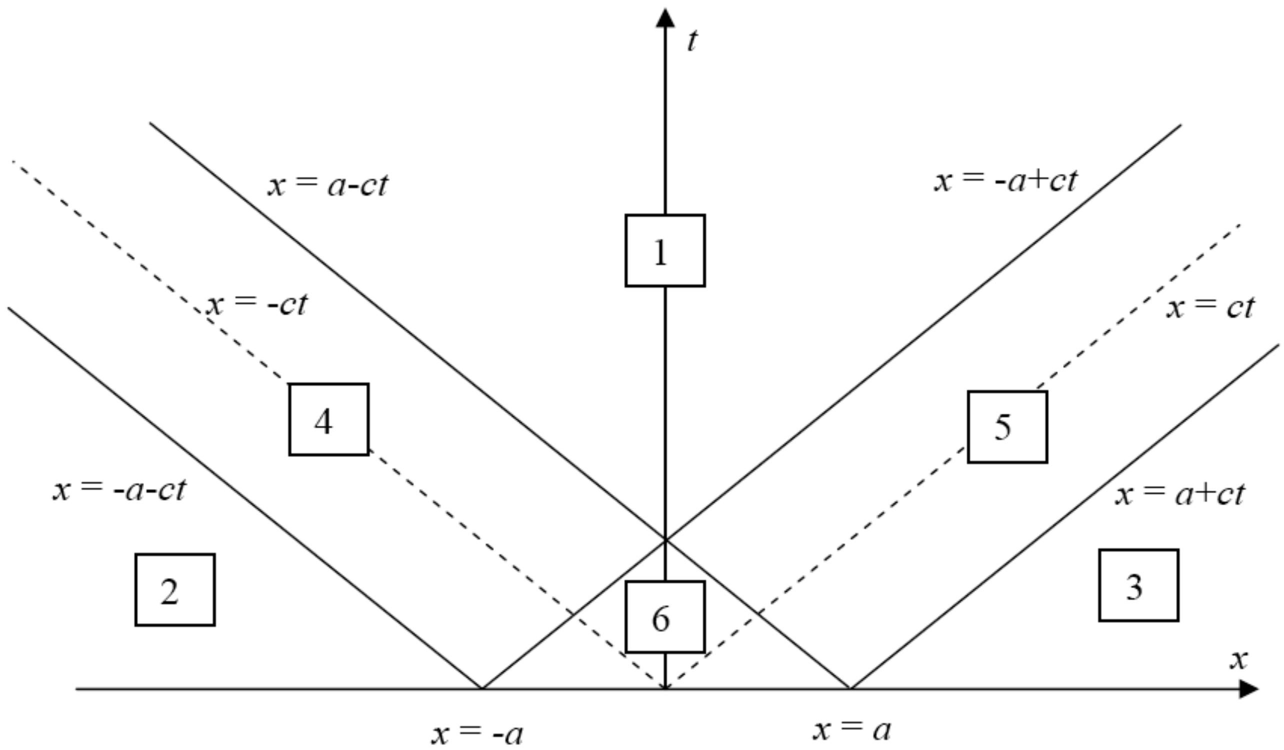
Ex. Wave with initial rectangular displacement released from rest, $V(x) = 0$

$$y(x,t) = \frac{1}{2}\left[U(x-ct) + U(x+ct)\right]$$

$$\left(y(x,t) = \frac{1}{2}\left[U(x-ct) + U(x+ct)\right] + \frac{1}{2c}\int_{x-ct}^{x+ct} V(x)\,dx \right)$$



This figure can also be represented on an (x,t) domain. Let $y(x,t)$ be pointing out of the paper.



In regions 1, 2, and 3, $y(x,t) = 0$ for all x, t

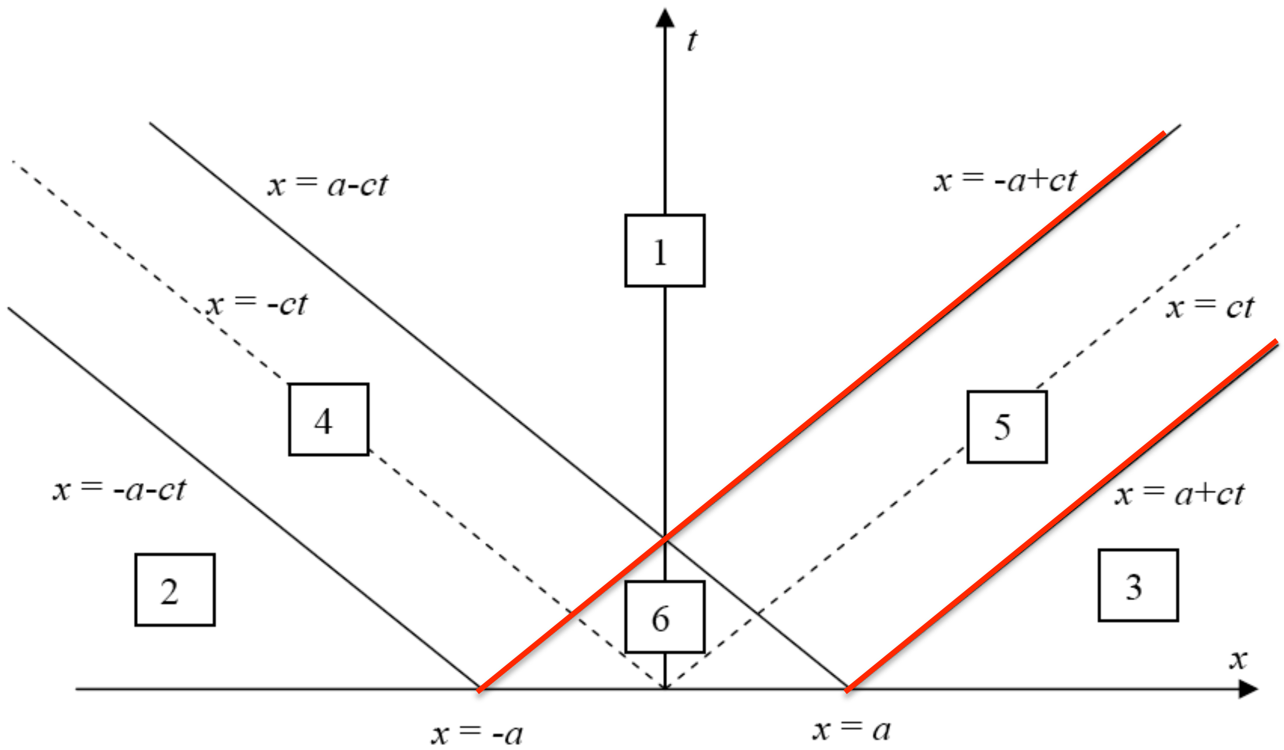
In region 5, $y(x,t) = \frac{1}{2}u(x-ct) \qquad -a \le x-ct \le a$

In region 4, $y(x,t) = \frac{1}{2}u(x+ct) \qquad -a \le x+ct \le a$

In region 6, $y(x,t) = \frac{1}{2}\left[u(x-ct)+u(x+ct)\right] \qquad x-ct > -a$

$$x+ct < a$$

This figure can also be represented on an (x,t) domain. Let $y(x,t)$ be pointing out of the paper.



In regions 1, 2, and 3, $y(x,t) = 0$ for all x, t

In region 5, $y(x,t) = \frac{1}{2}u(x-ct) \qquad -a \le x - ct \le a$

In region 4, $y(x,t) = \frac{1}{2}u(x+ct) \qquad -a \le x + ct \le a$

In region 6, $y(x,t) = \frac{1}{2}\left[u(x-ct) + u(x+ct)\right] \qquad x - ct > -a$

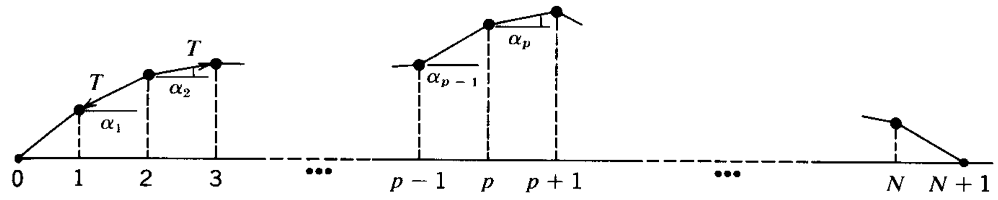
$x + ct < a$

$$y(x,t) = f(x - ct) + g(x + ct)$$

What is the form of $f(x)$, $g(x)$?

N coupled oscillators

Consider the transverse oscillations of N particles of mass m spaced equally along a flexible, elastic, massless string, which is under tension T .



Assume the particles are displaced by small distances y_i :

Consider the p^{th} particle

$$F_y = -T\sin\alpha_{p-1} + T\sin\alpha_p \approx -\frac{T}{l}\left(y_p - y_{p-1}\right) + \frac{T}{l}\left(y_{p+1} - y_p\right)$$

$$\ddot{y}_p + 2\omega_0^2 y_p - \omega_0^2\left(y_{p+1} - y_{p-1}\right) = 0, \qquad \omega_0^2 = T/ml$$

General solution :

Superposition of N normal modes

$$y_p = \sum_{n=1}^{N} \sin\left(\frac{pn\pi}{N+1}\right)\left(D_n \cos\left(\omega_n t + \delta_n\right)\right)$$

$$\omega_n = 2\omega_0 \sin\left[\frac{n\pi}{2\left(N+1\right)}\right]$$

$$y(x,t) = f(x-ct) + g(x+ct)$$

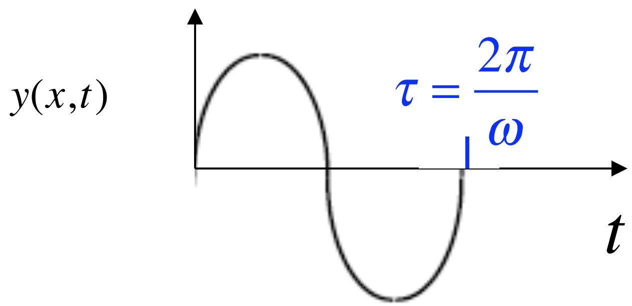
If time dependence is $\cos(\omega t)$ the full (x,t) dependence is given by

$$y(x,t) = A\cos(kx+\omega t) + B\cos(kx-\omega t)$$

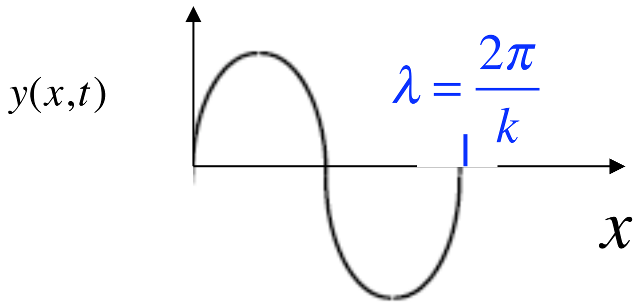
constant k

- Speed of wave $c = \dfrac{\omega}{k}$

- Frequency $f = \dfrac{1}{\tau} = \dfrac{\omega}{2\pi}$



- Wavelength $\lambda = \dfrac{2\pi}{\mathrm{k}}$



k is "wavenumber"

We can write the equation of a travelling wave in a number of analogous forms:

	Velocity	Wavelength	Period	Angular frequency
$A\sin(kx-\omega t)$	ω / k	$2\pi / k$	$2\pi / \omega$	ω
$A\sin k(x-vt)$	v	$2\pi / k$	$2\pi / vk$	vk
$A\sin\left[2\pi\left(\frac{x}{\lambda}-\frac{t}{\tau}\right)\right]$	λ / τ	λ	τ	$2\pi / \tau$
$A\sin\left[2\pi(x-vt)/\lambda\right]$	v	λ	λ / v	$2\pi v / \lambda$

N.B. Can include phase most easily by putting

$$y(x,t)=\mathrm{Re}\left[A\exp\left[i(kx-\omega t)\right]\right]$$

where A is complex.

N.B.2 Sometimes more convenient to switch x and t, i.e.

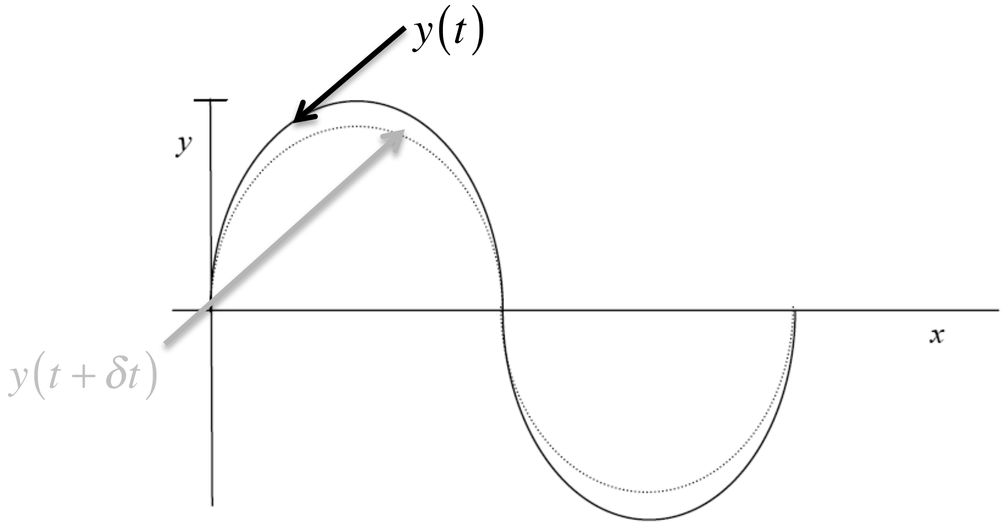
$$y(x,t)=A\sin(\omega t-kx)$$

This is still a travelling wave moving to the right.

For non-sinusoidal wave moving to right with speed v, can always write as $f(x-vt)$.

Stationary waves

$$y = A\sin(kx - \omega t) + A\sin(kx + \omega t)$$
$$= 2A\sin kx \cos \omega t$$



"stationary wave"

More generally

$$y = A\sin(kx - \omega t + 2\delta_1) + A\sin(kx + \omega t + 2\delta_2)$$
$$= 2A\sin(kx + \delta_1 + \delta_2)\cos(\omega t + \delta_1 - \delta_2)$$