# NORMAL MODES, WAVE MOTION AND THE WAVE EQUATION

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## Part 2    WAVE MOTION AND THE WAVE EQUATION


## 6 Introduction

The answer to the question "Why should we learn about waves" is simple: Waves are everywhere! For example

|                    |                              |
|--------------------|------------------------------|
| Strings            | Violin                       |
| Membranes          | Drum                         |
| Air                | Sound                        |
| Optics             | Interference and diffraction |
| E.M.               | Radio, T.V., …               |
| Quantum mechanics  | Uncertainty principle        |
|                    | α-decay                      |
| Seismology         | Earthquakes                  |

In the physics course you will study many of these so it is important that you should have a good understanding of the mathematical description of waves.

To start we will consider a simple case of a wave propagating along a string stretched along the x-axis. The string can have transverse vibrations corresponding to a displacement in the y-direction given by $y(x, t)$ at position x and time t. It is instructive first to consider a case similar to those we have been studying in which a system of masses on a stretched elastic string undergo transverse vibrations.

## 7   N coupled oscillators

Consider the transverse oscillations of $N$ particles of mass $m$ spaced equally along a flexible, elastic, massless string, which is under tension $T$.
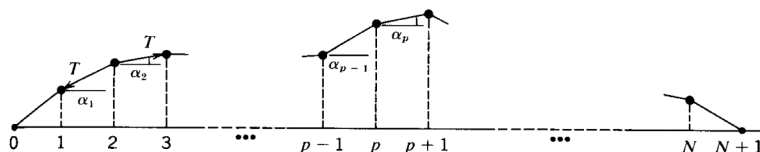


(reproduced  from French, 1971).

Assume the particles are displaced by small distances $y_i$ and thus the angles $\alpha_i$ are small too. In this case the length of the string between the particles is increased to $l' = l / \cos\alpha_i \approx l\left(1 + \alpha_i^2 / 2\right)$ i.e. $l' \approx l$ and the tension in the string remains constant.

Consider the $p^{\text{th}}$ particle above. The force acting in the $y$-direction is
$$F = -T\sin\alpha_{p-1} + T\sin\alpha_p \qquad (7.1)$$
which may be approximated by
$$F \approx -\frac{T}{l}\left(y_p - y_{p-1}\right) + \frac{T}{l}\left(y_{p+1} - y_p\right) \qquad (7.2)$$
Hence the equation of motion of the $p^{\text{th}}$ particle is
$$\ddot{y}_p + 2\omega_0^2 y_p - \omega_0^2\left(y_{p+1} - y_{p-1}\right) = 0 \qquad (7.3)$$
where $\omega_0^2 = T / ml$. We can write a similar equation for each of the $N$ particles and thus we have $N$ coupled differential equations and thus $N$ normal modes. The derivation of the solution is beyond the scope of these lectures so here we just quote the answer. The solution is a linear combination of the normal modes:
$$y_p = \sum_{n=1}^{N} \sin\left(\frac{pn\pi}{N+1}\right)\left(D_n \cos\omega_n t + E_n \sin\omega_n t\right) \qquad (7.4)$$
where
$$\omega_n = 2\omega_0 \sin\left[\frac{n\pi}{2(N+1)}\right] \qquad (7.5)$$

It is instructive to consider what these normal modes look like. For example for four coupled oscillators (N=4)
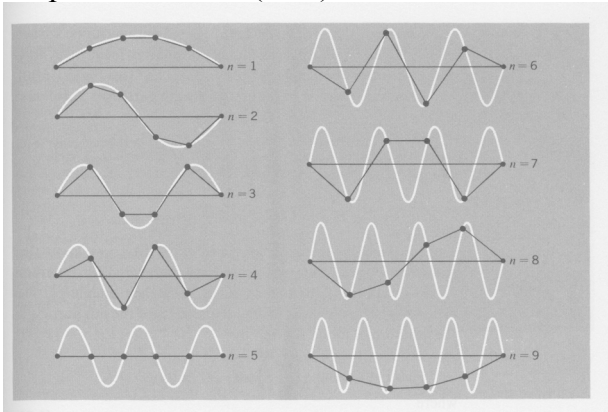


(reproduced from French, 1971)
Clearly there 4 normal modes in all. Note that $n = 6, 7, 8, 9$ repeat patterns of $n = 4, 3, 2, 1$ with opposite sign. This illustrates the start of the wave pattern, shown in the diagram in white, that we shall see occurs as $N \rightarrow \infty$ when the continuous distribution of masses becomes a string.

## 8 The wave equation – Transverse waves on a string

The simplest example of wave motion is that of transverse displacements of an elastic string. Consider the diagram of small portion of string ($\theta_1$ and $\theta_2$ small). Suppose the string has linear density (kg/m) $\rho$.
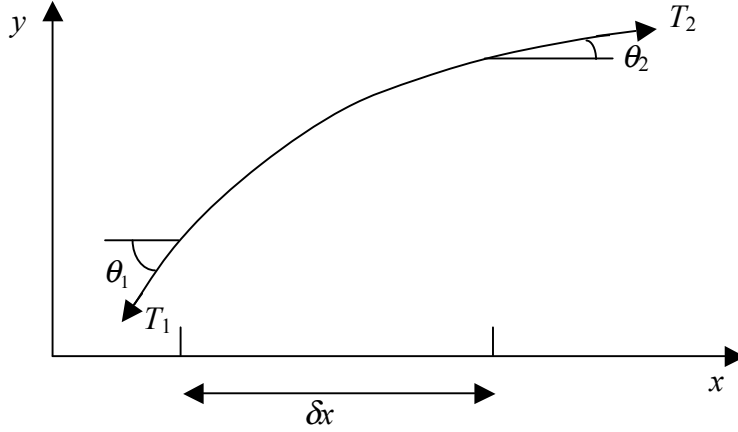


For small angle of displacement and transverse oscillations, as we discussed above, the tension, $T$, is approximately constant along the string. Resolving the forces acting on the portion of string in the y-direction we have, from Newton's second law

$$T\sin\theta_2 - T\sin\theta_1 = \left(\rho\delta x\right)\frac{\partial^2 y}{\partial t^2} \tag{8.1}$$

For small angles $\sin\theta \approx \tan\theta = \dfrac{\partial y}{\partial x}$ and hence

$$T\left[\left(\frac{\partial y}{\partial x}\right)_2 - \left(\frac{\partial y}{\partial x}\right)_1\right] = \rho\delta x\frac{\partial^2 y}{\partial t^2} \tag{8.2}$$

The final step is to replace $\dfrac{\partial y}{\partial x}$ by the leading terms in its Taylor series

$$\left(\frac{\partial y}{\partial x}\right)_2 = \left(\frac{\partial y}{\partial x}\right)_1 + \frac{\partial}{\partial x}\left(\frac{\partial y}{\partial x}\right)\delta x + \dots \tag{8.3}$$

Using this Eq.(8.2) becomes

$$T\left(\frac{\partial^2 y}{\partial x^2}\right)\delta x = \rho\frac{\partial^2 y}{\partial t^2}\delta x \tag{8.4}$$

and so we obtain the *Wave Equation*

$$\frac{\partial^2 y}{\partial x^2} = \frac{\rho}{T}\frac{\partial^2 y}{\partial t^2} \tag{8.5}$$

As we shall discuss this describes a wave moving with velocity $c = \sqrt{T/\rho}$
(hence larger tension or lighter string leads to faster waves).

## 9  D'Alembert's Solution

Following from Eq.(8.5) the wave equation is

$$\frac{\partial^2 y}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 y}{\partial t^2} \tag{9.1}$$

This is most easily solved by changing variables to

$$u = x - ct$$
$$v = x + ct \tag{9.2}$$

The wave equation may then be written in terms of these new variables by application of the chain rule. i.e. since $y(x,t) = y(u,v)$

$$\frac{\partial y}{\partial x} = \frac{\partial y}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial y}{\partial v}\frac{\partial v}{\partial x}$$
$$= \frac{\partial y}{\partial u} + \frac{\partial y}{\partial v} \tag{9.3}$$

similarly

$$\frac{\partial y}{\partial t} = \frac{\partial y}{\partial u}\frac{\partial u}{\partial t} + \frac{\partial y}{\partial v}\frac{\partial v}{\partial t}$$
$$= -c\frac{\partial y}{\partial u} + c\frac{\partial y}{\partial v} \tag{9.4}$$

Differentiating again we find:

$$\frac{\partial^2 y}{\partial x^2} = \frac{\partial^2 y}{\partial u^2} + 2\frac{\partial^2 y}{\partial u \partial v} + \frac{\partial^2 y}{\partial v^2} \tag{9.5}$$

and

$$\frac{\partial^2 y}{\partial t^2} = c^2\left[\frac{\partial^2 y}{\partial u^2} - 2\frac{\partial^2 y}{\partial u \partial v} + \frac{\partial^2 y}{\partial v^2}\right] \tag{9.6}$$

Hence substituting into the wave equation we find

$$\frac{\partial^2 y}{\partial u \partial v} = 0 \tag{9.7}$$

from which we may deduce that

$$y(u,v) = f(u) + g(v) \tag{9.8}$$

or

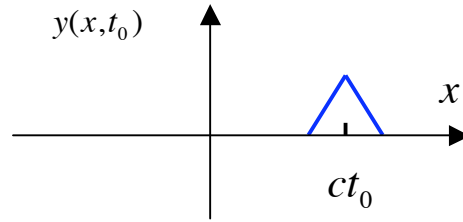$$y(x,t) = f(x - ct) + g(x + ct) \tag{9.9}$$

where $f$ and $g$ are *any* functions of $u$ and $v$. This is the general solution to the wave equation. The functions $f$ and $g$ are determined by the initial conditions as we shall show in Section 9.3. However first let us consider the meaning of this solution.

## 9.1 Travelling waves

Let us illustrate the solution just obtained by choosing $y(x,t) = f(x - ct)$ at $t = 0$ to be a pulse centered at the origin.
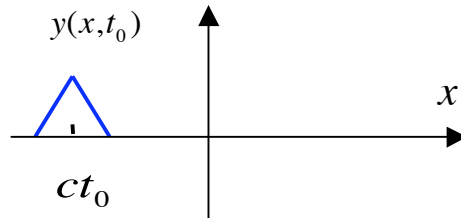
$y(x,0)$

Then at a later time $t = t_0$ the pulse remains the same shape but is translated to the right by a distance $ct_0$.
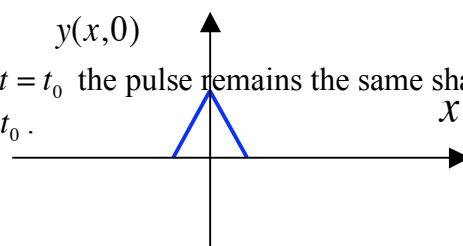
Thus $y(x,t) = f(x - ct)$ represents "travelling wave" moving to the right.

Now consider the case $y(x,t) = g(x + ct)$ at $t = 0$ to be a pulse centered at the origin as in the first diagram. Then at $t = t_0$ the pulse remains the same shape but is translated to the *left* by a distance $ct_0$.



Thus $y(x,t) = g(x + ct)$ represents "travelling wave" moving to the left.

### 9.2 D'Alembert's solution with boundary conditions

As we have just seen the functions $f$ and $g$ are determined by the initial conditions of the wave. These can be incorporated in d'Alembert's solution in a straightforward way. Suppose at time $t = 0$, the wave has an initial displacement $U(x)$ and an initial velocity $V(x)$

$$y(x,0) = f(x) + g(x) = U(x) \tag{9.10}$$

$$\frac{\partial y(x,0)}{\partial t} = -cf'(x) + cg'(x) = V(x) \tag{9.11}$$

integrating Eq. (9.11) gives:

$$f(x) - g(x) = -\frac{1}{c} \int_b^x V(x)\,dx \tag{9.12}$$

Adding Eqs (9.10) and (9.12) leads to

$$f(x) = \frac{1}{2}U(x) - \frac{1}{2c}\int_b^x V(x)\,dx \tag{9.13}$$

Subtracting Eqs. Eqs (9.10) and (9.12) leads to

$$g(x) = \frac{1}{2}U(x) + \frac{1}{2c}\int_b^x V(x)\,dx \qquad (9.14)$$

Hence combining these to form $y(x,t)$ we find:

$$y(x,t) = \frac{1}{2}\left[U(x-ct) + U(x+ct)\right] + \frac{1}{2c}\left[\int_b^{x+ct} V(x)\,dx - \int_b^{x-ct} V(x)\,dx\right] \qquad (9.15)$$
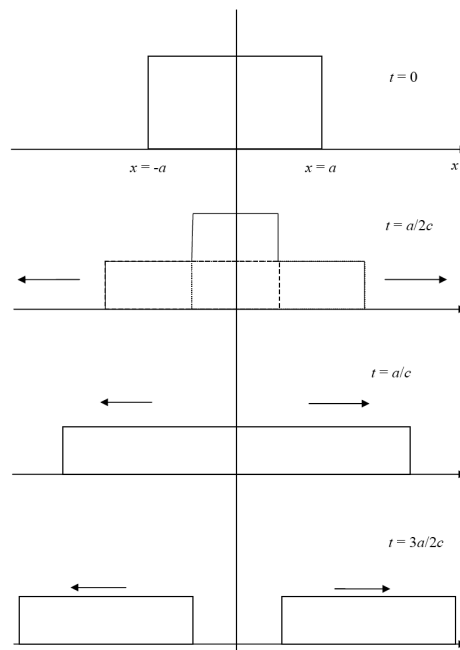
or

$$y(x,t) = \frac{1}{2}\left[U(x-ct) + U(x+ct)\right] + \frac{1}{2c}\int_{x-ct}^{x+ct} V(x)\,dx \qquad (9.16)$$
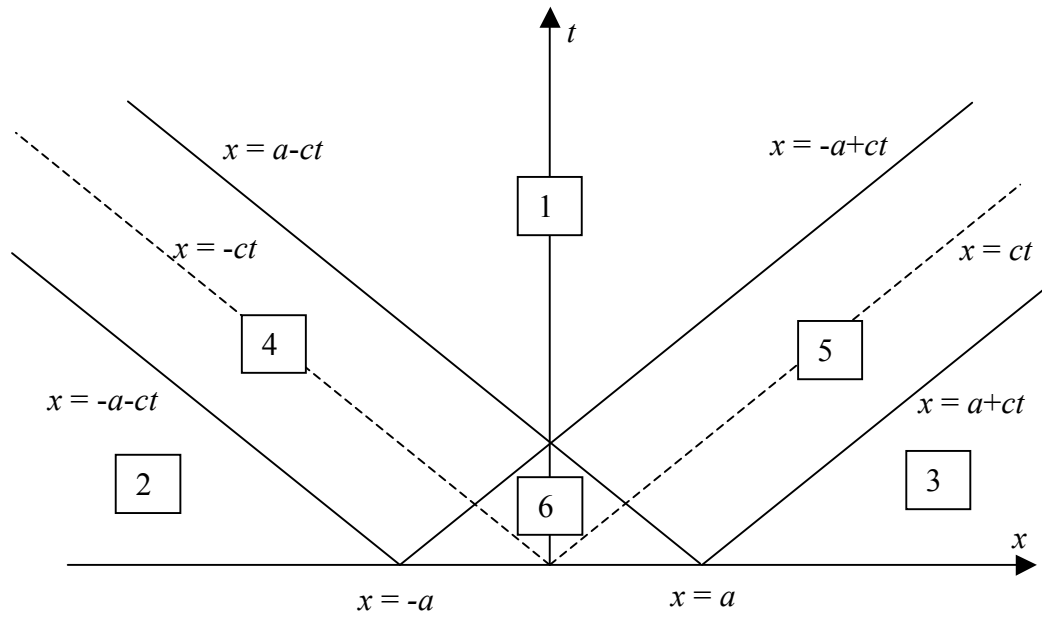
### 9.3 An example of D'Alembert's solution

A stretched string is released from rest (*i.e.* $V(x) = 0$) with an initial square displacement. Hence from Eq. (9.16)

$$y(x,t) = \frac{1}{2}\left[U(x-ct) + U(x+ct)\right] \qquad (9.17)$$

The resulting evolution is shown in the Figure below



This figure can also be represented on a space-time $(x,t)$ domain. Let $y(x,t)$ point out of the paper.

In regions 1, 2, and 3, $y(x,t) = 0$      for all $x$, $t$

In region 5, $y(x,t) = \dfrac{1}{2}u(x-ct)$      $-a \leq x - ct \leq a$

In region 4, $y(x,t) = \dfrac{1}{2}u(x+ct)$      $-a \leq x + ct \leq a$

In region 6, $y(x,t) = \dfrac{1}{2}\left[u(x-ct)+u(x+ct)\right]$      $x - ct > -a$

$x + ct < a$

## 10 Waves

### 10.1 Travelling waves

As we saw in Section 7 the case of the transverse oscillation of individual masses the time dependence of the normal modes oscillation is sinusoidal. Let us consider the case that the time dependence of the vibrating string at $x = 0$ is also sinusoidal, $y(x,0)=\sin(\omega t)$. In this case, from Eq. (9.9), the full $x,t$ dependence is given by the "wave"

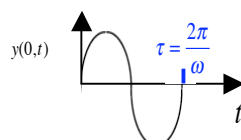$$y(x,t) = A\,\sin(kx + \omega t) + B\,\sin(kx - \omega t) \tag{10.1}$$

where A, B and $k$ are constants. The speed of the wave is

$$c = \frac{\omega}{k} \tag{10.2}$$

Its frequency, $f$, is inversely proportional to its period, $\tau$, and is given by

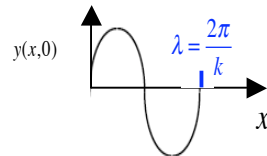$$f = \frac{1}{\tau} = \frac{\omega}{2\pi} \tag{10.3}$$

This is illustrated in the figure

Finally its wavelength, $\lambda$, is inversely proportional to its "wavenumber", $k$, and is given by

$$\lambda = \frac{2\pi}{k} \tag{10.4}$$

as is clear from the figure showing $y(x,0)$



We can write the equation of a travelling wave in a number of analogous forms:

| | Velocity | Wavelength | Period | Angular frequency |
|---|---|---|---|---|
| $A\sin(kx-\omega t)$ | $\omega / k$ | $2\pi / k$ | $2\pi / \omega$ | $\omega$ |
| $A\sin k(x-vt)$ | $v$ | $2\pi / k$ | $2\pi / vk$ | $vk$ |
| $A\sin\left[2\pi\left(\dfrac{x}{\lambda}-\dfrac{t}{\tau}\right)\right]$ | $\lambda / \tau$ | $\lambda$ | $\tau$ | $2\pi / \tau$ |
| $A\sin\left[2\pi(x-vt)/\lambda\right]$ | $v$ | $\lambda$ | $\lambda / v$ | $2\pi v / \lambda$ |

Note that it is often more convenient to represent a travelling wave by a complex exponential (this is particularly useful when one wants to combine phases):

$$y(x,t) = \mathrm{Re}\left\{A\exp\left[i(kx-\omega t)\right]\right\} = \mathrm{Re}\left\{|A|\exp\left[i(kx-\omega t+\phi)\right]\right\} \tag{10.5}$$

where $A$ is complex, $A = |A|e^{i\phi}$.

Sometimes it is more convenient to switch $x$ and $t$, i.e.

$$y(x,t) = A\sin(\omega t - kx) \tag{10.6}$$

This is still a travelling wave moving to the right.

Of course, as discussed above, for a non-sinusoidal wave moving to right with speed $c$, we can always write it as $f(x-vt)$ for some (non-sinusoidal) function $f$.

**10.2 Stationary waves**

Consider a string with two waves of equal amplitude travelling in the opposite directions

$$\begin{aligned} y &= A\sin(kx-\omega t) + A\sin(kx+\omega t) \\ &= 2A\sin kx \cos\omega t \end{aligned} \tag{10.7}$$

Consider now the displacement at some fixed $t$, e.g. $t = 0$, for which $y = 2A\sin kx$. Some small time $\delta t$ later the displacement becomes $y = 2A\cos(\omega.\delta t)\sin kx$. This has *exactly* the same $x$-dependence, and has *not* shifted at all (zeroes of $y$ stay at the same $x$) but the amplitude now just a bit smaller.
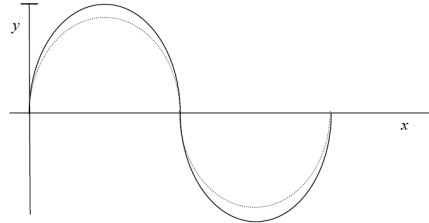


Hence as $t$ increases, the wave stays in the same place, but the amplitude varies.

e.g.

$$\text{at } \omega t = \pi/2 \qquad y = 0 \text{ everywhere}$$
$$\text{at } \omega t = \pi \qquad y(x,\pi/\omega) = -y(x,0)$$

This wave is called a "Stationary Wave". It can be written in several forms, *e.g.*
$y = 2A\sin\dfrac{2\pi x}{\lambda}\cos\dfrac{2\pi t}{\tau}$, and there may be arbitrary (constant) phases, *e.g.*
$2A\sin(kx + \phi_1)\cos(\omega t + \phi_2)$.


**10.3 Longitudinal and Transverse Waves and Polarisation**

A wave on a string has a displacement perpendicular to the (x) direction in which the wave travels *i.e.* it is a "Transverse" wave. But a sound wave has molecules moving backwards and forwards along the direction of wave propagation, *i.e.* sound waves are "Longitudinal" waves. Another example of a longitudinal wave is a coiled spring with the compression moving along spring.

Now clearly there is only one direction along the direction of propagation but there are two directions perpendicular to the direction of propagation *i.e.* a transverse wave moving along the $x-$axis can have two directions of transverse displacement, $y = A\sin(kx - \omega t)$ and $z = B\sin(kx - \omega t + \phi)$. Hence transverse waves can be "polarised", but longitudinal waves can not. Polarisation states correspond to definite values for the amplitudes A and B:

| $A$ | $B$ | $\phi$ | Polarisation state | |
|---|---|---|---|---|
| 1 | 0 | - | Linear | → |
| 0 | 1 | - | Linear | ↑ |
| 1 | 1 | 0 | Linear | ↗ |
| 1 | 1 | $\pi$ | Linear | ↖ |
| 1 | 1 | $\pi/2$ | Circular | (LH) |
| 1 | 1 | $-\pi/2$ | Circular | (RH) |

## 11 Group and Phase velocity.

### 11.1 Information transmission

Let us consider how a signal might be sent via a wave. It is necessary to modulate the wave otherwise it will convey zero information. An example is given by the signal shown below



Here the wave is *on* for time *T*, then *off* i.e. $y = A\sin(kx - \omega t)$ for $|kx - \omega t| \le \omega T / 2$ and $y = 0$ for $|kx - \omega t| > \omega T / 2$. It is important to note that this is *not* a single frequency wave for which $y = A\sin(kx - \omega_0 t)$ for some frequency $\omega_0$ and which would apply for **all** $kx - \omega_0 t$. In order to buld up a wave which is *off* at some time it is necessary to have a superposition of a range of frequencies.

[This actually requires FOURIER TRANSFORMS which we won't meet until the 2nd year, but we will give a simple illustration of the result below]

### 11.2 Group and phase velocity

Let us consider in more detail our example of a wave that conveys information which is made up of a superposition of many waves with a range of frequencies. For illustration we consider a superposition of a discrete number of waves.

$$y(x,t) = \sum_{n=1}^{N} D_n \cos(k_n x - \omega_n t) \qquad (11.1)$$

where $D_n$ are constants. An immediate question is how fast does envelope carrying the signal move? The answer to this is called the "group velocity". This can be quite different from the answer to the question how fast do the individual waves in the superposition travel? This is called the "phase velocity".
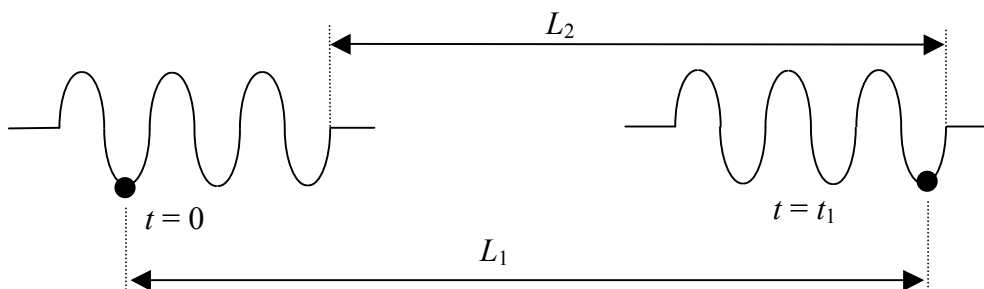
Let us first compute these velocities.



Figure 11.1

From Figure 11.1 we may determine the group velocity, $g$, by working out how far the front of the signal moves in time $t_1$. This is given by:

$$g = \frac{L_2}{t_1} \qquad (11.2)$$

What about the phase velocity? In Eq. (11.1) the individual waves have phase velocity $v_n = \frac{\omega_n}{k_n}$.

## Non-dispersive medium

If these are all equal, $v_n = v$, the envelope moves with unchanged shape and the phase velocity is the same as the group velocity. That this is the case follows from the fact that in this case Eq. (11.1) may be rewritten as

$$y(x,t) = \sum_{n=1}^{N} D_n \cos\left(k_n x - \omega_n t\right) = \sum_{n=1}^{N} D_n \cos k_n \left(x - vt\right) = y(x - vt) \qquad (11.3)$$

i.e. a function of $(x - vt)$ only. This is the form of d'Alembert's solution obtained in Section 9.2 and shows that the whole wave moves with the phase velocity, v, with unchanged shape, i.e. the group velocity equals the phase velocity.

## Dispersive medium

Now let us consider waves in a "dispersive medium", *i.e.* one in which different frequencies are transmitted at different speeds:

$$y = A\sin\left(kx - \omega t\right) \text{ where } v = \omega / k = f\left(\omega\right) \qquad (11.4)$$

A well known example is the passage of light through a glass prism where different colours emerge at different angles because the refractive index $\mu \, (= c/v)$ depends on the colour ($\omega$), i.e. v depends on $\omega$.

What happens to our example in Eq. (11.1)? In this case the individual phase velocities $v_n = \frac{\omega_n}{k_n}$ are not all equal and the group velocity can be quite different from the phase velocity. This is illustrated in Figure 11.1 where one phase velocity, given by $v_p = \frac{L_1}{t_1}$, is greater than the group velocity $g$. To make this clearer let us turn to a more definite, albeit oversimplified, model of a wave packet of the form of Eq. (11.1) but with $N = 2$, i.e. a superposition of just two waves.

## 11.4 A simple approach to building a wave packet

Although we really need an infinite number of different frequency waves to construct a finite wave packet here we will illustrate the result using just two waves, $y_1$ and $y_2$.

$$y_1 = A\sin\left[\left(k + \delta k\right)x - \left(\omega + \delta\omega\right)t\right]$$
$$y_2 = A\sin\left[\left(k - \delta k\right)x - \left(\omega - \delta\omega\right)t\right] \qquad (11.5)$$

where $\delta k$ and $\delta\omega$ are small. Then

$$y = y_1 + y_2 = 2A\cos\left(\delta k.x - \delta\omega.t\right)\sin\left(kx - \omega t\right) \qquad (11.6)$$

The first term has a very long wavelength $2\pi/\delta k$ (and very long period $2\pi/\delta\omega$). This describes the slowly varying envelope which moves at speed $g = \delta\omega/\delta k$.

The second term is very similar to $y_1$ and $y_2$ and has speed $v = \omega/k$ for the individual wavelets.

The addition of these waves gives the following
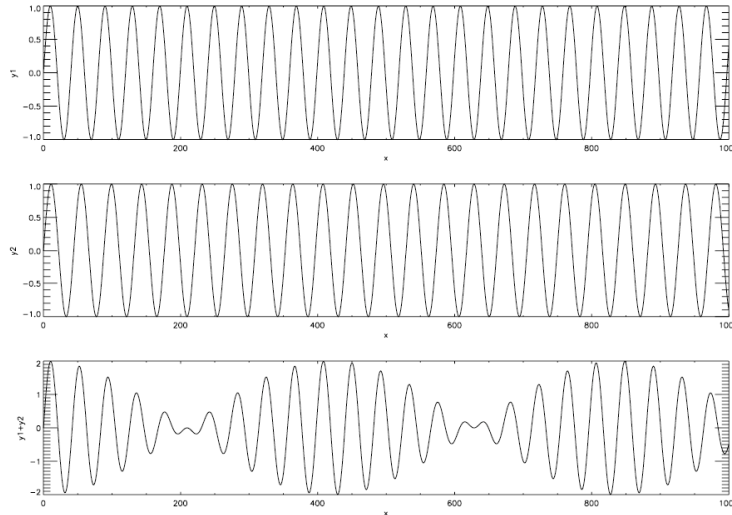


*Plot for $y_1, y_2$ and $y_1 + y_2$ where $\delta k / k = 0.05$.*

This is not exactly one packet of waves, but an infinite series of sausages (because we used two waves instead of infinite number). However the essential point is that envelope moves with speed $g = \delta\omega/\delta k$ and not at the mean phase velocity $v = \omega/k$.

## 11.5 Group velocity for a complete wave packet

For the case of a real wave packet of finite extent comprised of an infinite number of different frequency waves the group velocity is given by

$$g = \frac{d\omega}{dk} \qquad (11.7)$$

For the case of a dispersive medium with $v = \omega/k = f(\omega)$ then $\dfrac{d\omega}{dk}$ need not equal

$\dfrac{\omega}{k}$ and so the group velocity need not equal the phase velocity. As we have discussed it is the group velocity that determines the speed a signal can propagate and this is constrained by the theory of relativity to be less than or equal to the speed of light , $g \leq c$. However the phase velocity, $v$, can readily be greater than $c$.

We can illustrate this with a simple example. Consider an experiment to determine the speed of light by measuring the time of flight of a pulse of light through a long tube filled with air to determine its velocity, $v_0$. Since $\mu \neq 1$, we need to correct the

measured time to determine the speed of light, $c$, in vacuum. The naïve answer is $c = v_0 \, \mu$ but this does not give the group velocity if $\mu$ depends on $k$ i.e. on the frequency.

To make this explicit let us assume the phase velocity , $v$, varies with colour as

$v = \dfrac{\omega}{k} = c\left(1 - bk\right)$, *i.e.* $\mu = \dfrac{c}{v} = \dfrac{1}{1 - bk}$ . Then $\omega = c\left(k - bk^2\right)$ giving for the group

velocity $g = \dfrac{\partial \omega}{\partial k} = c\left(1 - 2bk\right)$, different from the phase velocity, $v = \dfrac{c}{\mu}$ . From this one

sees it is incorrect to change the measured velocity $v_0$ to $v_0 \, \mu$ to allow for the effect of the air.

## 11.6 Dispersion and the spreading of the wave packet

Since a finite wave packet involves a superposition of waves with a range of frequencies, if the velocity is frequency dependent then, necessarily, the shape of the packet will change because the individual wave components are moving with
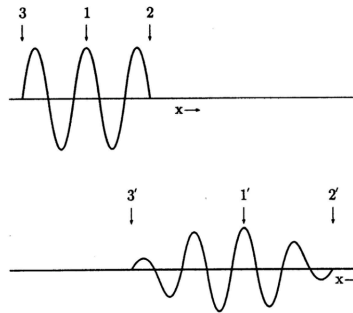


different speed. This is illustrated above. However this immediately demonstrates a problem to measuring the group velocity because if the shape of the wave packet is changing it is impossible precisely to measure the envelope's speed. This effect may

be seen algebraically from the fact that the group velocity $g = \dfrac{d\omega}{dk}$ is in general a

function of the frequency and thus not uniquely determined for a wave packet comprised of a range of frequencies. In practice this may not be so important for a long wave packet as in this case the range of frequencies in the packet is quite small so the ambiguity of the group velocity is also small.

## 11.7 Alternative expressions for *g*

There are many equivalent expressions for the group velocity. Above we used

$$g = \frac{d\omega}{dk} \tag{11.8}$$

But $\omega = vk$ so we can write it as

$$g = v + k\frac{dv}{dk} \tag{11.9}$$

Further, since $k = 2\pi / \lambda$ we also have

$$g = v - \lambda\frac{dv}{d\lambda} \tag{11.10}$$

Yet another form is given using $v = c / \mu$

$$g = \frac{c}{\mu}\left(1 + \frac{\lambda}{\mu}\frac{d\mu}{d\lambda}\right)$$

(11.11)

*i.e.* $g \neq c / \mu$ in a dispersive medium as we found in our example above.

Note here we have used $k$ and $\lambda$ as measured in a medium. More conventionally we use the wavelength, $\lambda'$, measured in vacuum. In terms of it we have

$$g = v\left(1 - \frac{1}{1 + \frac{v}{\lambda'}\frac{d\lambda'}{dv}}\right)$$

since $\lambda' = \lambda\dfrac{c}{v}$ and $f = \dfrac{c}{\lambda'} = \dfrac{v}{\lambda}$.

## 12 Energy of vibrating string

Let us assume the string carries a transverse wave $y = A\sin(kx - \omega t)$ and consider a small portion of the string as shown in the figure below. The string has linear density $\rho$. There are two contributions to the energy, the kinetic energy and the potential energy.
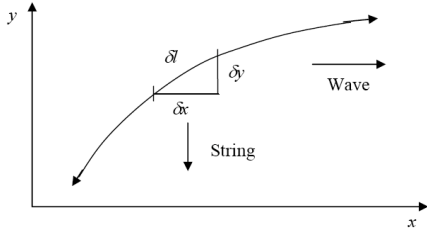
**Kinetic energy (K.E.)**

The K.E. of the segment shown is given by

$$K.E. = \frac{1}{2}\rho\delta x\left(\frac{\partial y}{\partial t}\right)^2$$

$$= \frac{1}{2}\rho A^2\omega^2\cos^2(kx - \omega t)\delta x$$

(12.1)

We may now integrate over a distance $l$ that contains a whole number of wavelengths, at fixed $t$.

$$K.E. = \frac{1}{2}\rho A^2\omega^2\int_x^{x+l}\cos^2(kx - \omega t)dx = \frac{1}{2}\rho A^2\omega^2 \times \frac{l}{2}$$

(12.2)

Hence the K.E. per unit length is given by

$$K.E./l = \frac{1}{4}\rho A^2 \omega^2 \qquad (12.3)$$

**Potential energy**

The wave stretches the string leading to an increase in its potential energy relative to the value in its equilibrium position. The potential energy in a the segment is given by the work done in stretching the string. The tension, T, is the force resisting the stretching so we have

$$P.E. \text{ of segment } = T(\delta l - \delta x)$$

$$= T\delta x \left( \sqrt{1 + \left(\frac{\delta y}{\delta x}\right)^2} - 1 \right) \qquad (12.4)$$

$$\approx \frac{1}{2}TA^2 k^2 \cos^2(kx - \omega t)\delta x$$

Integrating this over $l$ leads to the result

$$P.E./l = \frac{1}{4}TA^2 k^2 \qquad (12.5)$$

How do the K.E. and P.E. compare? Since $v = \omega / k = \sqrt{T/\rho}$ we have $Tk^2 = \rho\omega^2$ and hence

$$P.E./l = \frac{1}{4}TA^2 k^2 = \frac{1}{4}\rho A^2 \omega^2 = K.E./l \qquad (12.6)$$

Hence the total energy per unit length is given by

$$E/l = \frac{1}{2}\rho A^2 \omega^2 \qquad (12.7)$$

**12.1 Energy flow**

In time $t$, the wave moves a distance $vt$ (note that since we are considering a plane wave of definite frequency it is the phase velocity, v, that is relevant). Hence the energy flow/unit time, F, is given by

$$F = \left(\frac{1}{2}\rho A^2 \omega^2\right)vt / t$$

$$= \frac{1}{2}\rho\omega^2 A^2 v$$

$$= \frac{1}{2}\rho\omega^3 A^2 / k \qquad (12.8)$$

$$= \frac{1}{2}Tk^2 A^2 v$$

$$= \frac{1}{2}T\omega k A^2$$

( Note that the energy flow may also be calculated by considering the rate at which the string to the left of a position does work on the right.

$$E = F_y v_y$$

$$= -T \frac{\partial y}{\partial x} \frac{\partial y}{\partial t}$$

(12.9)

Substituting for $y$ and differentiating we find the same answer.)

## 13 Solution of the wave equation – separation of variables

In this section we introduce the method of separation of variables for finding solutions to the wave equation. In Section 8 we derived the wave equation which for convenience we repeat here

$$\frac{\partial^2 y}{\partial x^2} = \frac{1}{v^2} \frac{\partial^2 y}{\partial t^2}$$

(13.1)

where v is the speed of the wave.

We now look for solutions that have the "separated" form:

$$y(x,t) = X(x)T(t)$$

(13.2)

Substituting this into the wave equation we find:

$$T(t)\frac{d^2 X(x)}{dx^2} = \frac{1}{v^2} X(x)\frac{d^2 T(t)}{dt^2}$$

(13.3)

or

$$\frac{\ddot{X}}{X} = \frac{1}{v^2}\frac{\ddot{T}}{T}$$

(13.4)

Now the left-hand side of this equation is a function of just $x$, while the right-hand side is a function of just $t$. The only way that a function of $x$ can equal a function of $t$ for all $x$ and $t$ is if both are equal to a *constant*, $C_S$. We look for a solution in which this constant, known as the "separation constant", is *negative*, $C_S = -k^2$. i.e.

$$\frac{\ddot{X}}{X} = \frac{1}{v^2}\frac{\ddot{T}}{T} = C_S = -k^2$$

(13.5)

Then we find that

$$\ddot{X} + k^2 X = 0$$
$$\ddot{T} + k^2 v^2 T = 0$$

(13.6)

The separation constant reduces the partial differential equation to two ordinary differential equations which may be solved using standard methods to give:

$$X = A\cos kx + B\sin kx$$
$$T = C\cos kvt + D\sin kvt$$

(13.7)

where $A$, $B$, $C$ and D are unknown constants which may be found from the boundary conditions. If the boundary conditions constrain $A = 0$, and $D = 0$ then we find that

$$y(x,t) = X(x)T(t) = B'\sin kx \cos kvt$$

(13.8)

(B'=BC) which is a standing wave of the form that we introduced earlier in Eq. (10.7) (N.B. $kv=\omega$). Eq. (10.7) shows the connection of the solution we have just obtained by separation of variables with that we obtained previously by d'Alembert's solution.

In deriving this solution we chose our separation constant to be negative. What happens if we had chosen it to be positive? If $C_S = +k^2$ then we find:

$$y(x,t) = X(x)T(t) = \left(Ae^{kx} + Be^{-kx}\right)\left(Ce^{kvt} + De^{-kvt}\right) \qquad (13.9)$$

If $C_S = 0$ then we find:

$$y(x,t) = X(x)T(t) = (A + Bx)(C + Dt) \qquad (13.10)$$

and the list continues for complex separation constants. Which of these solutions is elevant depends on the physical situation. In these lectures we are concerned with the sinusoidally varying solutions that correspond the negative choice for the separation variable.

Even having chosen the sign the solution is not unique because any value of $k$ may be used. Indeed, since the differential equation is linear, the principle of superposition applies and a linear combination of any number of solutions with different values of $k$ will still be a solution. In the next section we will see how this can be used to find general solutions to various physical situations.

## 13.1 Wave on a string with fixed ends



$x = 0$        $x = \mathrm{L}$
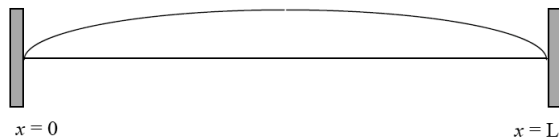
Consider a string given an initial displacement, $y(x,0)$, and then released. What happens subsequently? Since the string has fixed ends we also have the boundary conditions $y(0,t) = y(L,t) = 0$.

We have found by separation of variables that a solution to the wave equation is given by:

$$y(x,t) = (A\cos kx + B\sin kx)(C\cos kvt + D\sin kvt) \qquad (13.11)$$

As we shall see we can use a linear superposition of such solutions with different choices for $k$ to build a solution that satisfies the boundary conditions. Let us consider the boundary conditions in turn.

    i)      The string is initially at rest, *i.e.* $\partial y / \partial t = 0$ for all $x$, which requires $D = 0$
    ii)     $y(0,t) = 0$ requires that $A = 0$
    iii)    $y(L,t) = 0$ requires that $kL = n\pi$, where $n$ is any integer. This latter condition is known as the eigenvalue equation and it limits $k$ to be an integer multiple of $n\pi / L$ ; each of these values corresponds to a normal mode.

With this we can write the most general solution consistent with the boundary conditions i)-iii) as a linear superposition of the normal mode solutions:

$$y(x,t) = \sum_{n=0}^{\infty} A_n \sin\frac{n\pi x}{L}\cos\frac{n\pi vt}{L} \qquad (13.12)$$

So far we have not imposed the final initial condition, namely the initial displacement $y(x,0)$. If the initial string displacement initially corresponds to a normal mode

$$y(x,0) = B\sin\frac{m\pi x}{L} \tag{13.13}$$

then by comparision with Eq. (13.12), $A_n = B$ for $n = m$, $A_n = 0$ otherwise and the subsequent motion is given by

$$y(x,t) = B\sin\frac{m\pi x}{L}\cos\frac{m\pi vt}{L}$$

What if initial displacement does not correspond to a normal mode?
e.g. a plucked guitar string with shape
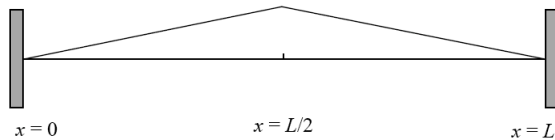


In this case the subsequent motion described by an infinite sum of normal modes as in Eq. (13.12). At $t = 0$

$$y(x,0) = \sum_n A_n \sin\frac{n\pi x}{L} \tag{13.14}$$

and the form of the initial displacement will determine the $A_n$ coefficients. In the second year you will study Fourier series of this type. It turns out that the coefficients can easily be found due to the orthogonality of the sine functions over the range $0 \le x < L$, giving

$$A_n = \frac{2}{L}\int_0^L y(x,0)\sin\frac{n\pi x}{L}dx \tag{13.15}$$

However, since this lies beyond the scope of these lectures, let us consider a simple case in which the initial distribution is given by

$$y(x,0) = \sin\frac{\pi x}{L} + \frac{1}{2}\sin\frac{2\pi x}{L} \tag{13.16}$$

Comaring with Eq. (13.12) we see that at subsequent times:

$$y(x,t) = \sin\frac{\pi x}{L}\cos\frac{\pi vt}{L} + \frac{1}{2}\sin\frac{2\pi x}{L}\cos\frac{2\pi vt}{L} \tag{13.17}$$

Note that in this case we have a superposition of two normal modes and, unlike the case with just a single normal mode, the subsequent motion is not equal to the initial displacement $\times$ varying amplitude. Moreover since the shorter wavelengths oscillate faster the shape of the wave varies during oscillation.


## 14 Wave reflection at a boundary

In optics reflection is caused at a boundary separating regions with different refractive indices in which the light travels at different speeds. Exactly the same phenomena occurs for the transverse waves propagating on a string. In this case on obtains a boundary separating regions with different wave speeds by joining two strings of

different linear densities, $\rho_{1,2}$. Since the tension, $T$, remains the same across the boundary, the phase velocities are different

$$v_{1,2} = \sqrt{T / \rho_{1,2}} \qquad (14.1)$$

This is illustrated in the diagram



$\rho_1$       $\rho_2$

$x = 0$
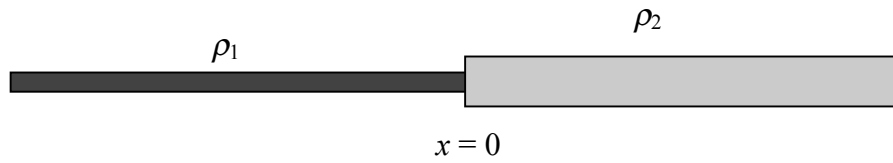
| $x < 0$ | $x > 0$ |
|---|---|
| Lighter string, larger v | Heavier string, smaller v |
| $\rightarrow$ Incident wave | $\rightarrow$ Transmitted wave |
| $\leftarrow$ Reflected wave | |

| Incident wave | $A\sin(\omega t - k_1 x)$ |
|---|---|
| Reflected wave | $A'\sin(\omega t + k_1 x)$ |
| Transmitted wave | $A''\sin(\omega t - k_2 x)$ |

Note:
1) We have made a slight change in the convention using $\omega t - kx$ instead of $kx - \omega t$. This is not crucial, but agrees with other treatments.
2) All waves have the same $\omega$ [This follows from the boundary conditions at $x = 0$. These cannot be satisfied for all $t$ unless $\omega$ is constant – see below]
3) The transmitted wave has $-k_2 x$ (right mover)
4) The reflected wave has $+k_1 x$ (left-mover)

The amplitudes $A'$ and $A''$ are determined in terms of the incident amplitude $A$ from the boundary conditions at $x = 0$. There are two boundary conditions

A) $y(-\varepsilon, t) = y(+\varepsilon, t)$ where $\varepsilon$ is a number close to zero. This just says the string is continous.

B) $\dfrac{\partial y}{\partial x}(-\varepsilon, t) = \dfrac{\partial y}{\partial x}(+\varepsilon, t)$. This follows because, for small angular displacements, the vertical component of the force on the left of the boundary $T\dfrac{\partial y}{\partial x}(-\varepsilon, t)$ must be balanced by the vertical component of the force on the left of the boundary $T\dfrac{\partial y}{\partial x}(\varepsilon, t)$. (For small displacements the horizontal component of the force vanishes to the order considered here – see the discussion above Eq. (7.1))

These boundary conditions are more commonly written as

$$y_1(0,t) = y_2(0,t)$$
$$\frac{\partial y_1}{\partial x}(0,t) = \frac{\partial y_2}{\partial x}(0,t) \qquad (14.2)$$

where $y_1(x,t) = A\sin(\omega t - k_1 x) + A'\sin(\omega t + k_1 x)$ is the sum of the incident and reflected waves on the left of the boundary and $y_2(x,t) = A''\sin(\omega t - k_2 x)$ is the transmitted wave on the right of the boundary. Thus we have

$$A\sin\omega t + A'\sin\omega t = A''\sin\omega t \implies A + A' = A'' \qquad (14.3)$$

and

$$-k_1 A\cos\omega t + k_1 A'\cos\omega t = -k_2 A''\cos\omega t \implies k_1(A - A') = k_2 A'' \qquad (14.4)$$

These may be solved to give

$$r = \frac{A'}{A} = \frac{k_1 - k_2}{k_1 + k_2}$$

$$t = \frac{A''}{A} = \frac{2k_1}{k_1 + k_2}$$

(14.5)

where $r$ and $t$ are known as the *amplitude reflection* and *amplitude transmission* coefficients respectively.

**Special cases**:

1) $k_1 = k_2 \qquad \implies A' = 0, \; t = \dfrac{A''}{A} = 1 \qquad$ No reflection

2) $k_1 < k_2 \qquad \implies A'$ is negative
In this case the reflected wave may be written as
$-|A'|\sin(\omega t + k_1 x) = |A'|\sin(\omega t + k_1 x + \pi)$ i.e. there is a phase change at a **rare-dense** boundary (since $v = \omega / k = \sqrt{T / \rho}$, $k_1 < k_2$ implies $\rho_1 < \rho_2$).

3) $k_1 > k_2 \qquad \implies A'$ is positive

4) $\rho_2 \to \infty \qquad \implies k_2 \to \infty$ hence $r = \dfrac{A'}{A} \to -1$. In this case the tension $T \to 0$ and there is no wave in the very heavy string.


**14.1 The energy flux at the boundary**

In Eq. (12.8) we showed that the energy flux is proportional to the (amplitude)$^2$. It is thus very tempting, but **wrong** to think that the power reflection and transmission coefficients should be just $R_P = r^2$, and $T_P = t^2$. That this cannot be true can immediately be seen from the fact that $R_P + T_P \neq 1$ which would correspond to non-conservation of energy. To see this, note that from Eq. (14.5)

$$r^2 + t^2 = \left(\frac{k_1 - k_2}{k_1 + k_2}\right)^2 + \left(\frac{2k_1}{k_1 + k_2}\right)^2 = \frac{5k_1^2 - 2k_1 k_2 + k_2^2}{(k_1 + k_2)^2} \neq 1$$

The origin of the error is clear from Eq.(12.8) which expresses the power flux as

$P = \dfrac{1}{2}T\omega k A^2$. Although $T$ and $\omega$ are constant within the string $k$ is different on each side of the boundary, explaining the error. Hence at the boundary between the two strings:

Incident power flux
$$P_I = \frac{1}{2}T\omega k_1 A^2$$

Reflected power flux
$$P_R = \frac{1}{2}T\omega k_1 A'^2$$

Transmitted power flux
$$P_T = \frac{1}{2}T\omega k_2 A''^2$$

Hence the power reflection and transmission coefficients are given by

$$R_P = \frac{P_R}{P_I} = \frac{A'^2}{A^2} = r^2 = \left(\frac{k_1 - k_2}{k_1 + k_2}\right)^2 \tag{14.6}$$

$$T_P = \frac{P_T}{P_I} = \frac{k_2 A''^2}{k_1 A^2} = \frac{k_2}{k_1}t^2 = \frac{k_2}{k_1}\left(\frac{2k_1}{k_1 + k_2}\right)^2 = \frac{4k_1 k_2}{\left(k_1 + k_2\right)^2} \tag{14.7}$$
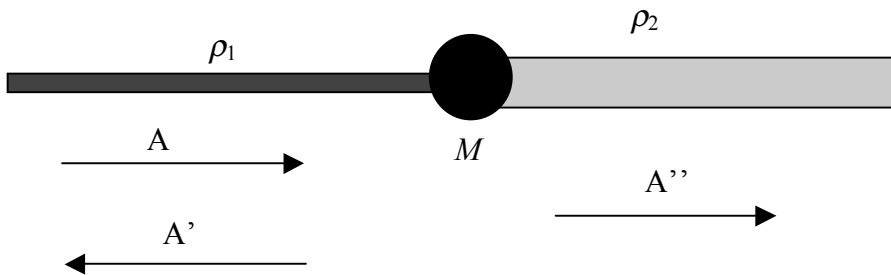
Hence

$$R_P + T_P = \left(\frac{k_1 - k_2}{k_1 + k_2}\right)^2 + \frac{k_2}{k_1}\left(\frac{2k_1}{k_1 + k_2}\right)^2 = \frac{k_1^2 + 2k_1 k_2 + k_2^2}{\left(k_1 + k_2\right)^2} = 1 \tag{14.8}$$

as required by energy conservation.

## 14.2 Reflection from a mass at the boundary

Suppose that a finite mass $M$ is fixed at the boundary between two semi-infinite pieces of string of density $\rho_1$ and $\rho_2$:



The string is clearly continuous and hence the first boundary condition is
$$y_1(0,t) = y_2(0,t) \tag{14.9}$$
as before. The second boundary condition however is not the same since we now have a finite mass at the boundary. In this case the sum of the forces at the boundary act on the mass and generates its acceleration in the transverse direction according to Newton's 2nd law:

$$-T\frac{\partial y_1}{\partial x}(0,t) + T\frac{\partial y_2}{\partial x}(0,t) = M\frac{\partial^2 y_1}{\partial t^2}(0,t) = M\frac{\partial^2 y_2}{\partial t^2}(0,t) \tag{14.10}$$

Eqs. (14.9) and (14.10) are the boundary conditions for this problem. In this case we see that they involve both first and second derivatives and for this reason it is easiest to use a complex exponential representation for the waves i.e.

$$y_1(x,t) = \text{Re}\left\{A\exp\left(i\left(\omega t - k_1 x\right)\right)\right\} + \text{Re}\left\{A'\exp\left(i\left(\omega t + k_1 x\right)\right)\right\}$$

$$y_2(x,t) = \text{Re}\left\{A''\exp\left(i\left(\omega t - k_2 x\right)\right)\right\}$$

(14.11)

where $A$ is real but $A'$ and $A''$ may be complex. Inserting this in Eqs. (14.9) and (14.10) gives

$$A + A' = A''$$

(14.12)

and

$$ik_1 TA - ik_1 TA' - ik_2 TA'' = -\omega^2 M\left(A + A'\right) = -\omega^2 MA''$$

(14.13)

which simplifies to

$$ik_1\left(A - A'\right) = \left(ik_2 - \omega^2 M / T\right)A''$$

(14.14)

From Eqs. (14.12) and (14.14) we can determine the amplitude reflection and amplitude transmission coefficients:

$$r = \frac{A'}{A} = \frac{\left(k_1 - k_2\right)T - i\omega^2 M}{\left(k_1 + k_2\right)T + i\omega^2 M} \equiv |r|e^{i\phi_r}$$

(14.15)

$$t = \frac{A''}{A} = \frac{2k_1 T}{\left(k_1 + k_2\right)T + i\omega^2 M} \equiv |t|e^{i\phi_t}$$

(14.16)

Substituting this in Eq. (14.11) gives the real amplitudes with the reflected and transmitted waves having phase shifts $\phi_r$ and $\phi_t$ relative to the incident wave.

Consider the special case where the second line has zero mass per unit length, i.e. we just have a mass on the end of a line. i.e. $k_2 = 0$. Then:

$$r = \frac{A'}{A} = \frac{k_1 T - i\omega^2 M}{k_1 T + i\omega^2 M}$$

(14.17)

Hence if $M = 0$, $r = 1$ and if $M$ is large, $r = -1 = e^{i\pi}$.

## 15 Characteristic Impedance

Although more commonly used for cases of electromagnetic waves travelling in transmission lines or space, the concept of "characteristic Impedance" may actually be defined for *any* wave motion and is a useful descriptive parameter.

For transverse waves on a string the characteristic impedance $Z$ is defined as the force acting in the y-direction divided by the velocity of the string in the y-direction, *i.e.*

$$Z = \frac{F_y}{v_y} = \frac{-T\dfrac{\partial y}{\partial x}}{\dfrac{\partial y}{\partial t}} \tag{15.1}$$

For a sine wave travelling in the positive $x$-direction $y(x,t) = A\sin(kx - \omega t)$ and thus the characteristic impedance is:

$$Z = \frac{Tk}{\omega} = \frac{T}{v} = (T\rho)^{1/2}$$

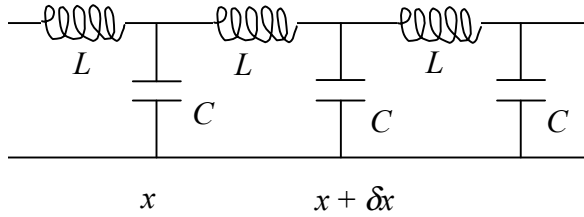We may express the reflection and transmission coefficients of Eqs. (14.15) and (14.16) in terms of impedances

$$r = \frac{A'}{A} = \frac{(k_1 - k_2)T - i\omega^2 M}{(k_1 + k_2)T + i\omega^2 M} = \frac{Z_1 - (Z_2 + Z_M)}{Z_1 + (Z_2 + Z_M)}$$

$$t = \frac{A''}{A} = \frac{2k_1 T}{(k_1 + k_2)T + i\omega^2 M} = \frac{2Z_1}{Z_1 + Z_2 + Z_M} \tag{15.2}$$

where we have substituted the wavenumbers for the characteristic impedances $Z_{1,2} = \dfrac{Tk_{1,2}}{\omega}$ and also for the 'impedance' of the mass: $Z_M = i\omega M$.

## 16 Other Waves

### 16.1 Waves on an Electrical Line



The voltage change across the inductor of self-inductance $L$ in one of the elements is given (via Faraday's law) as:

$$L\frac{\partial I}{\partial t} = -\delta V = -\frac{\partial V}{\partial x}\delta x \tag{16.1}$$

where we have assumed that the elements are so small that the voltage change can be related to the variation of voltage with distance.

Similarly the current flowing though the capacitance $C$ in each element is given by:

$$\frac{\partial Q}{\partial t} = C\frac{\partial V}{\partial t} = -\delta I = -\frac{\partial I}{\partial x}\delta x \tag{16.2}$$

Hence dividing these two equations by $\delta x$ gives:

$$\frac{L}{\delta x}\frac{\partial I}{\partial t} = L'\frac{\partial I}{\partial t} = -\frac{\partial V}{\partial x} \qquad (16.3)$$

and

$$\frac{C}{\delta x}\frac{\partial V}{\partial t} = C'\frac{\partial V}{\partial t} = -\frac{\partial I}{\partial x} \qquad (16.4)$$

where $C'$ is the capacitance per unit length of the transmission line and $L'$ is the self-inductance per unit length.

Differentiating Eq. (16.3) with respect to $t$ and differentiating Eq. (16.4) with respect to $x$ gives:

$$\frac{\partial^2 V}{\partial t \partial x} = -\frac{1}{C'}\frac{\partial^2 I}{\partial x^2} = -L'\frac{\partial^2 I}{\partial t^2} \qquad (16.5)$$

Hence:

$$\frac{\partial^2 I}{\partial x^2} = L'C'\frac{\partial^2 I}{\partial t^2} \qquad (16.6)$$

which is a wave equation for $I$. Similarly we can obtain a wave equation for $V$:

$$\frac{\partial^2 V}{\partial x^2} = L'C'\frac{\partial^2 V}{\partial t^2} \qquad (16.7)$$

Hence the electrical line supports waves with a phase speed $v = 1/\sqrt{L'C'}$.

Suppose we have a voltage wave travelling in a single direction: $V = V_0 \sin(\omega t - kx)$, then using the above equations we find that $I = (V_0 / Z)\sin(\omega t - kx)$ where $Z$ is the characteristic impedance given by $Z = \sqrt{L'/C'}$.

**Reflection at a terminated line**
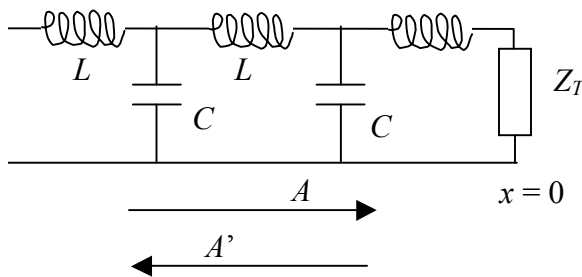


Voltage wave travels along a transmission line of characteristic impedance $Z_0$ and is partly reflected by a terminating impedance $Z_T$ at x = 0. The voltage and current on the line are thus given by:

$$\begin{aligned} V &= A\exp\left(i\left(\omega t - kx\right)\right) + A'\exp\left(i\left(\omega t + kx\right)\right) \\ Z_0 I &= A\exp\left(i\left(\omega t - kx\right)\right) - A'\exp\left(i\left(\omega t + kx\right)\right) \end{aligned} \qquad (16.8)$$

At x = 0, $V$ and $I$ are related by the terminating impedance and thus:

$$\frac{V}{Z_0 I} = \frac{Z_T}{Z_0} = \frac{A + A'}{A - A'} \qquad (16.9)$$

Hence

$$r = \frac{A'}{A} = \frac{Z_T - Z_0}{Z_T + Z_0} \qquad (16.10)$$

Hence

when $\quad Z_T \to 0, \quad r \to -1$
when $\quad Z_T = Z_0, \quad r = 0$
when $\quad Z_T \to \infty, \quad r \to +1$

These limits appear to be the reverse of what we found for the mass on the end of a string. However the characteristic impedance is defined differently. For strings $Z = Tk / \omega = T / \text{v}$. However for transmission lines $Z = L'\text{v}$ which is somewhat different !

## 16.2 Sound

Sound waves correspond to longitudinal waves associated with the compression of the medium. Consider waves propagating in the x-direction as shown in the Figure
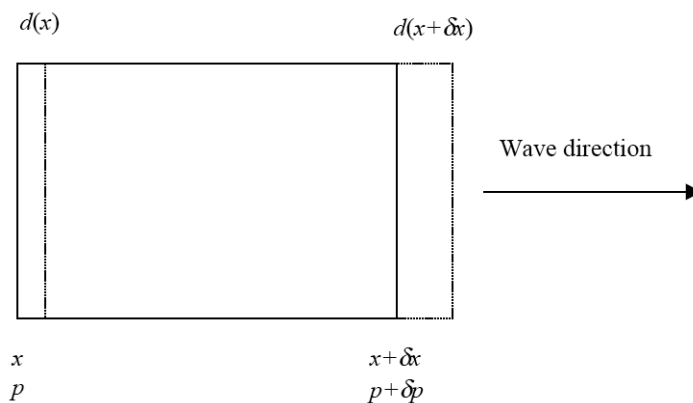


The compression caused by the passage of the sound wave changes the pressure and the volume of the element of gas originally between $x$ and $x+\delta x$. Consider a cross-sectional area $A$ of the wave.

With no sound wave, the volume of the element of gas is $V_1 = A\delta x$. With the sound wave passing the new volume of the element is

$$V_2 = A\left(d\left(x + \delta x\right) - d\left(x\right)\right) + V_1$$
$$\approx A\frac{\partial d}{\partial x}\delta x + V_1 \qquad (16.11)$$
$$= V_1 \frac{\partial d}{\partial x} + V_1$$

Hence

$$\delta V = V_2 - V_1 = \frac{\partial d}{\partial x} V \qquad (16.12)$$

The instantaneous pressure of the gas may be approximated by

$$p \approx p_0 + \frac{\partial p}{\partial V} \delta V$$

$$= p_0 + \frac{\partial p}{\partial V} \frac{\partial d}{\partial x} V \qquad (16.13)$$

$$= p_0 - K \frac{\partial d}{\partial x}$$

where $K$ is the bulk modulus of the gas (or solid) defined as:

$$K = -V \frac{\partial p}{\partial V} \qquad (16.14)$$

The net force, F, acting on the element is:

$$F = A \left[ p(x) - p(x + \delta x) \right]$$

$$= -\frac{\partial p}{\partial x} A \delta x$$

$$= K \frac{\partial^2 d}{\partial x^2} A \delta x \qquad \left( p = p - K \frac{\partial d}{\partial x} \right) \qquad (16.15)$$

$$= K \frac{\partial^2 d}{\partial x^2} V$$

From Newton's 2$^{nd}$ law

$$F = \frac{\partial^2 d}{\partial t^2} \rho V \qquad (16.16)$$

Hence, comparing these equations

$$\frac{\partial^2 d}{\partial x^2} = \frac{\rho}{K} \frac{\partial^2 d}{\partial t^2} \qquad (16.17)$$

which is the wave equation with $v = \sqrt{\dfrac{K}{\rho}}$ .

The characteristic impedance is defined as

$$Z = -\frac{K \dfrac{\partial d}{\partial x}}{\dfrac{\partial d}{\partial t}} \qquad (16.18)$$

so for a wave travelling in positive $x$-direction

$$Z = \frac{Kk}{\omega} = \frac{K}{v} = \left( \rho K \right)^{1/2} \qquad (16.19)$$

Isothermal compressions:  $PV = \text{constant} \Rightarrow K = -V \frac{\partial p}{\partial V} = p \Rightarrow v = \sqrt{p / \rho}$

Adiabatic compressions:  $PV^\gamma = \text{constant} \Rightarrow K = -V \frac{\partial p}{\partial V} = \gamma p \Rightarrow v = \sqrt{\gamma p / \rho}$

From kinetic theory we know $p = \dfrac{1}{3}\rho\overline{v^2}$ where here v is the molecular speed. Hence

$v_{sound} = \sqrt{\dfrac{\gamma}{3}\overline{v^2}}$ and thus $v_{sound} \approx v_{rms}$ of molecules since sound is transmitted by moving molecules.

N.B. For longitudinal waves travelling along solid bars, we get very similar solutions except that the bulk modulus $K$ must be replaced by Young's modulus $Y$.