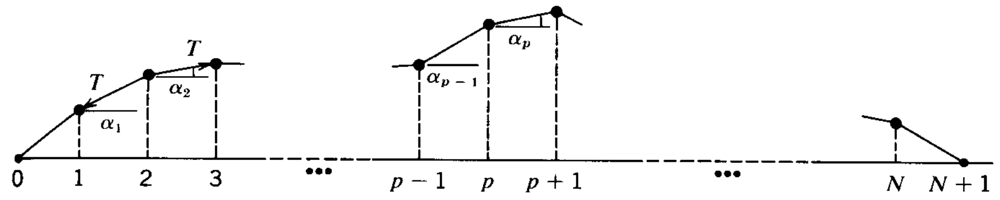
Waves

Waves are everywhere :

Strings	Violin
Membranes	Drum
Air	Sound
Optics	Geometrical optics, interference and diffraction
E.M.	Radio, T.V.,….
Quantum Mechanics	Wave/particle, uncertainty principle, ß decay…
Seismology	Earthquakes

N coupled oscillators

Consider the transverse oscillations of N particles of mass m spaced equally along a flexible, elastic, massless string, which is under tension T.



Assume the particles are displaced by small distances y_i :

α_i small

$\sin\alpha_i \simeq \alpha_i \simeq \tan\alpha_i$

$\cos\alpha_i \simeq 1-\frac{1}{2}\alpha_i^2 \simeq 1$

$$l' = l / \cos\alpha \approx l \quad \Rightarrow \quad \text{tension in string constant, T}$$

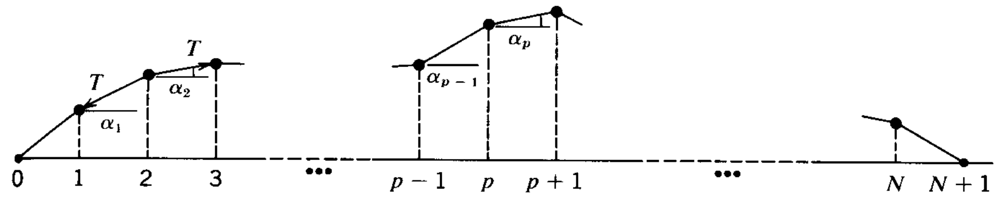
$$F_x = -T\cos\alpha_{p-1} + T\cos\alpha_p \simeq 0$$

$$F_y = -T\sin\alpha_{p-1} + T\sin\alpha_p \approx -\frac{T}{l}\left(y_p - y_{p-1}\right) + \frac{T}{l}\left(y_{p+1} - y_p\right) = m\ddot{y}_p$$

$$\ddot{y}_p + 2\omega_0^2 y_p - \omega_0^2\left(y_{p+1} - y_{p-1}\right) = 0, \qquad \omega_0^2 = T / ml$$

N coupled oscillators

Consider the transverse oscillations of N particles of mass m spaced equally along a flexible, elastic, massless string, which is under tension T.



Assume the particles are displaced by small distances y_i :

Consider the p^{th} particle

$$F_y = -T\sin\alpha_{p-1} + T\sin\alpha_p \approx -\frac{T}{l}\left(y_p - y_{p-1}\right) + \frac{T}{l}\left(y_{p+1} - y_p\right)$$

$$\ddot{y}_p + 2\omega_0^2 y_p - \omega_0^2\left(y_{p+1} - y_{p-1}\right) = 0, \qquad \omega_0^2 = T / ml$$

General solution :

Superposition of N normal modes

$$y_p = \sum_{n=1}^{N} \sin\left(\frac{pn\pi}{N+1}\right)\left(D_n \cos\left(\omega_n t + \delta_n\right)\right)$$

$$\omega_n = 2\omega_0 \sin\left[\frac{n\pi}{2\left(N+1\right)}\right]$$

E.G. N=2

$$\ddot{y}_p + 2\omega_0^2 y_p - \omega_0^2\left(y_{p+1} - y_{p-1}\right) = 0, \qquad \omega_0^2 = T / ml$$

$$\ddot{y}_1 + 2\omega_0^2 y_1 - \omega_0^2\left(y_2\right) = 0,$$

$$\ddot{y}_2 + 2\omega_0^2 y_2 - \omega_0^2\left(y_1\right) = 0$$

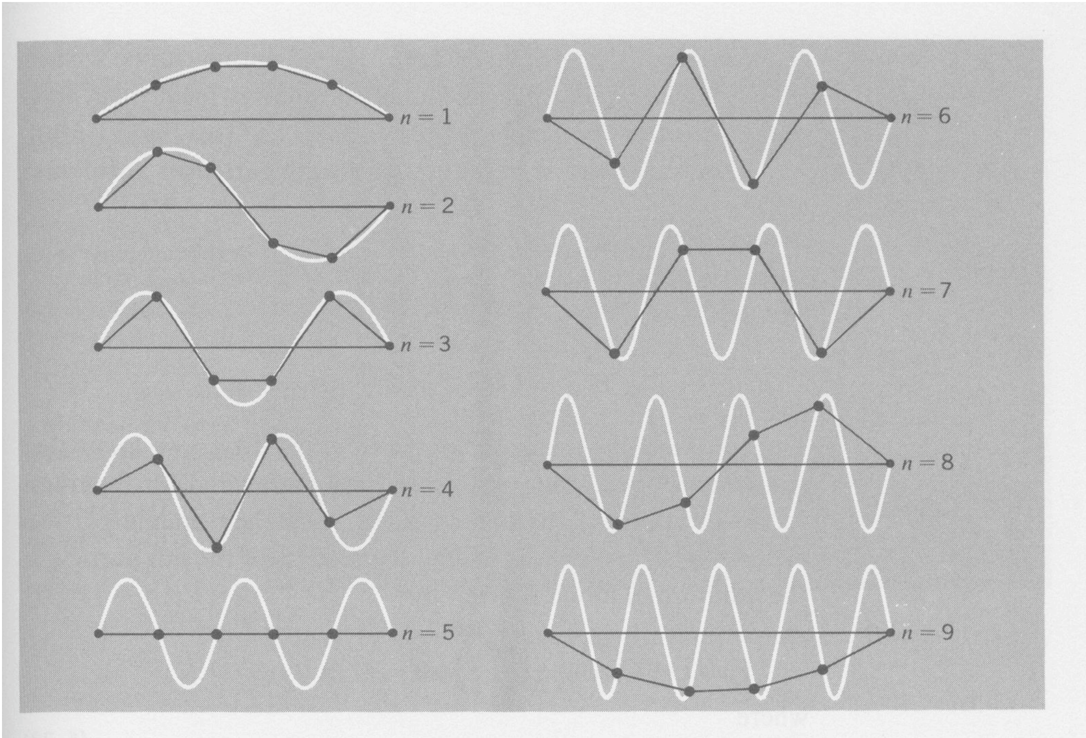
$$\ddot{q}_1 + \omega_0^2 q_1 = 0 \qquad q_1 = y_1 + y_2$$

$$\ddot{q}_2 + 3\omega_0^2 q_2 = 0 \qquad q_2 = y_1 - y_2$$

$$\left(c.f.\ y_p = \sum_{n=1}^{N} \sin\left(\frac{pn\pi}{N+1}\right)\left(D_n \cos\left(\omega_n t + \delta_n\right)\right), \quad \omega_n = 2\omega_0 \sin\left[\frac{n\pi}{2\left(N+1\right)}\right], N = 2 \right)$$

E.G. N=4

$$y_p = \sum_{n=1}^{N} \sin\left(\frac{pn\pi}{N+1}\right)\left(D_n \cos\omega_n t + E_n \sin\omega_n t\right), \quad \omega_n = 2\omega_0 \sin\left[\frac{n\pi}{2(N+1)}\right], \quad N=4$$



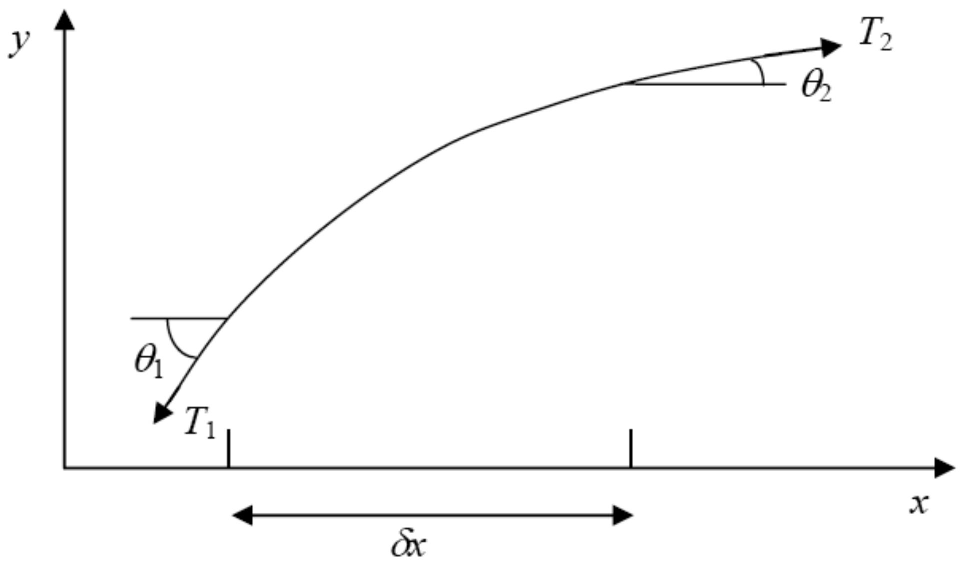
(reproduced from French, 1971)

Clearly there 4 normal modes in all. Note that n = 6, 7, 8, 9 repeat patterns of n = 4, 3, 2,1 with opposite sign.

In the limit $N \to \infty$ one obtains the wave equation for transverse waves

- The Wave Equation

Transverse displacements of an elastic string of linear density (kg/m) ρ



Resolve horizontal forces : $T_1 \cos\theta_1 = T_2 \cos\theta_2$ for small θ, $\cos\theta \sim 1 \Rightarrow T_1 = T_2 = T$

Resolve vertical forces $T\sin\theta_2 - T\sin\theta_1 = (\rho\delta x)\dfrac{\partial^2 y}{\partial t^2}$

$$\therefore T\left[\left(\frac{\partial y}{\partial x}\right)_2 - \left(\frac{\partial y}{\partial x}\right)_1\right] = \rho\delta x \frac{\partial^2 y}{\partial t^2}$$

$\left(\sin\theta \approx \tan\theta = \dfrac{\partial y}{\partial x}\right)$

$$T\left[\left(\frac{\partial y}{\partial x}\right)_2 - \left(\frac{\partial y}{\partial x}\right)_1\right] = \rho\delta x \frac{\partial^2 y}{\partial t^2}$$

$$\left(\frac{\partial y}{\partial x}\right)_2 = \left(\frac{\partial y}{\partial x}\right)_1 + \frac{\partial}{\partial x}\left(\frac{\partial y}{\partial x}\right)\delta x + \ldots \quad \Rightarrow \quad T\left(\frac{\partial^2 y}{\partial x^2}\right)\delta x = \rho \frac{\partial^2 y}{\partial t^2}\delta x$$

$$\Rightarrow \quad \frac{\partial^2 y}{\partial x^2} = \frac{\rho}{T}\frac{\partial^2 y}{\partial t^2} \equiv \frac{1}{c^2}\frac{\partial^2 y}{\partial t^2}$$

This is a WAVE EQUATION with velocity $c = \sqrt{T / \rho}$
(hence larger tension or lighter string leads to faster waves)

D'Alembert's solution

$$\frac{\partial^2 y}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 y}{\partial t^2}$$

Change variables

$$u = x - ct$$
$$v = x + ct$$

$$\frac{\partial}{\partial x} = \frac{\partial u}{\partial x}\frac{\partial}{\partial u} + \frac{\partial v}{\partial x}\frac{\partial}{\partial v} = \frac{\partial}{\partial u} + \frac{\partial}{\partial v}$$

$$\frac{\partial}{\partial t} = \frac{\partial u}{\partial t}\frac{\partial}{\partial u} + \frac{\partial v}{\partial t}\frac{\partial}{\partial v} = -c\frac{\partial}{\partial u} + c\frac{\partial}{\partial v}$$

$$\frac{\partial^2 y}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 y}{\partial t^2} \Rightarrow \left(\frac{\partial}{\partial u} + \frac{\partial}{\partial v}\right)^2 y = \frac{1}{c^2}\left(-c\frac{\partial}{\partial u} + c\frac{\partial}{\partial v}\right)^2 y = \left(\frac{\partial}{\partial u} - \frac{\partial}{\partial v}\right)^2 y$$

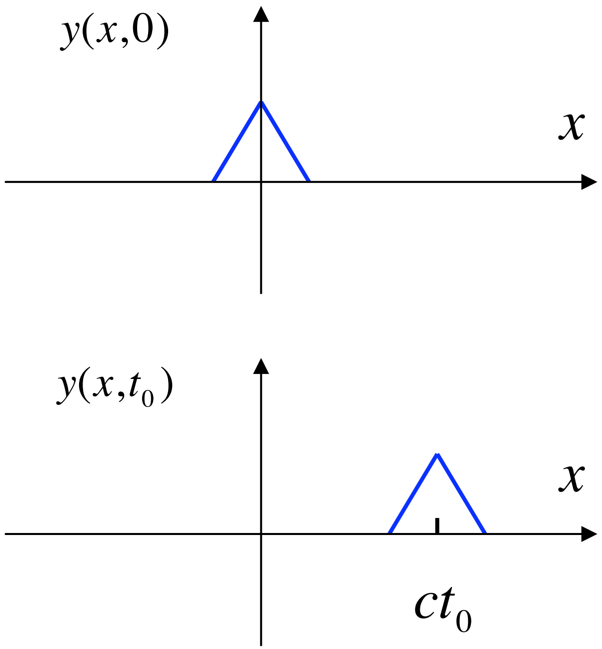
$$\Rightarrow \boxed{\frac{\partial^2 y}{\partial u \partial v} = 0}$$

f, g arbitrary functions

General solution $y(u,v) = f(u) + g(v)$ *i.e.* $y(x,t) = f(x - ct) + g(x + ct)$

Travelling waves

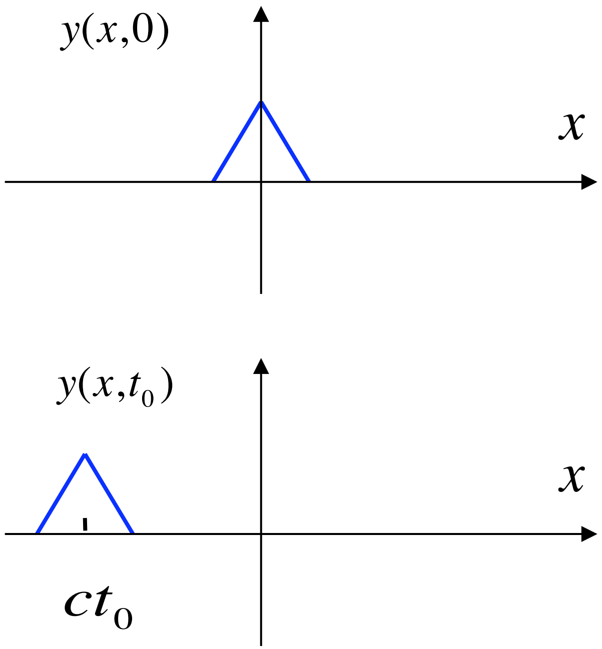
$$y(x,t) = f(x - ct)$$



Wave moves to right with speed c

Travelling waves

$$y(x,t) = g(x+ct)$$



Wave moves to left with speed c