Graduate Particle Physics MT 2005

Scattering Theory Problem Set

1 Find the Born approximation to the differential cross section as a function of the angle of scattering for a central potential $V(r) = -V_0$ for $r \leq r_0$, $V(r) = 0$ for $r > r_0$.

2 The potential for 'hard sphere' scattering is $V(r) = \infty$ for $r < r_0$ and $V(r) = 0$ for $r > r_0$. (i) Show that the s-wave phase shift $\delta_0(k)$ is given by $\delta_0(k) = -kr_0$, and calculate the total $\ell = 0$ cross section.

(ii) The 'reduced' radial wave function $u_\ell(r)$ is defined via $\psi(r, \theta, \phi) = R_\ell(r)\, Y_{\ell,m}(\theta, \phi)$, and $R_\ell(r) = u_\ell(r)/r$. Find the equation satisfied by u_ℓ, and show that for $r > r_0$ and $\ell = 1$ it is satisfied by

$$u_1(r) = C\left[\frac{\sin kr}{kr} - \cos kr + a\left(\frac{\cos kr}{kr} + \sin kr\right)\right]$$

where C and a are constants. Show that the definition of the p-wave phase shift $\delta_1(k)$ implies that $\tan\, \delta_1(k) = a$, and determine a from the condition satisfied by $u_1(r)$ at $r = r_0$.

Show that, as k approaches zero, $\delta_1(k)$ behaves like $(kr_0)^3$.

3 The scattering amplitude $f(\theta, k)$ has the partial wave expansion

$$f(\theta, k) = \frac{1}{k}\sum_\ell e^{i\delta_\ell} \sin \delta_\ell (2\ell + 1) P_\ell(\cos\theta)$$

where the symbols have their usual meaning.

(i)Show

(a) $\sigma_{tot} = \sum_\ell \sigma_\ell$ where $\sigma_\ell = \frac{4\pi}{k^2}\sin^2 \delta_\ell.(2\ell + 1)$

(b) $\sigma_{tot} = \frac{4\pi}{k}\mathrm{Im} f(0, k)$.

(ii) In the 'Fermi theory' of lepton-lepton scattering the s-wave amplitude $a_0 \equiv e^{i\delta_0}\sin\delta_0/k$ for the scattering process

$$\bar{\nu}_\mu + \mu^- \rightarrow \bar{\nu}_e + e^-$$

is given approximately by $a_0^F \approx G_F k$ where k is the centre of mass momentum and G_F is Fermi's constant, with the value (in units $\hbar = c = 1$)

$$G_F \approx \frac{10^{-5}}{M_p^2}$$

where M_p is the proton mass. Assuming that k in part (i) can also be taken to be the CM momentum (in units $c = \hbar = 1$) estimate the CM energy (neglecting lepton masses) at which unitarity is violated in the theory. What actually happens? (Note: $\hbar \approx 200$(MeV/c)fm).

Take $\hbar = c = 1$ for the remainder of this question

(iii) According the the Standard Model of particle physics, Higgs bosons should interact with each other in the $\ell = 0$ partial wave via an amplitude a_0^H which is of order $\frac{m_H^2 G_F}{k}$ in magnitude, where m_H is the Higgs mass.

Estimate the upper limit on the Higgs mass.

(iv) The width Γ for the decay Higgs $\rightarrow W^+ + W^-$ is given, for $m_H >> m_W$, by

$$\Gamma = G_F {m_H}^3 / 8\pi\sqrt{2}.$$

At what value of m_H will this width be comparable to m_H itself, and what would this signify?

(v) The 'Breit-Wigner' resonance cross-section is given by

$$\sigma_{res}(E) = \sigma(E_R)\,\frac{\Gamma^2/4}{(E - E_R)^2 + k^2\Gamma^2/4}$$

where E is the CM energy of the particles forming (or emitted by) the resonance, k is the CM momentum, E_R is the resonance energy and Γ is a width parameter. Sketch $\sigma_{res}(E)$ as a function of E. Also sketch $\sigma_{res}(E)$ for the case (which is approximately that of the Z^0) in which $\Gamma = \gamma k^2$, where γ is a constant, assuming that $\gamma k_R^3 << E_R$.

4 In first order time-dependent perturbation theory, the transition probability per unit time for a particle in the momentum eigenstate $\mid \boldsymbol{k}_i \rangle$ to scatter, via the time-independent potential $V(r)$, into the momentum eigenstate $\mid \boldsymbol{k}_f \rangle$ is

$$\int 2\pi \mid \langle \boldsymbol{k}_f \mid V \mid \boldsymbol{k}_i \rangle \mid^2 \delta(E_i - E_f) \frac{d^3\mathbf{k}_f}{(2\pi)^3}$$

where units $\hbar = 1$ are used (for a concise derivation see P79, Quarks and Leptons, Halzen & Martin, Wiley 1984). Writing

$$d^3\mathbf{k}_f = \mid \mathbf{k}_f \mid^2 d \mid \mathbf{k}_f \mid d\Omega \quad \text{and} \quad E_f = \mathbf{k}_f^2/2m$$

show that $d^3\mathbf{k}_f = m \mid \mathbf{k}_f \mid dE_f d\Omega$, and hence show that the differential cross section is

$$\frac{d\sigma}{d\Omega} = \frac{m^2}{4\pi^2} \mid \langle f \mid V \mid i \rangle \mid^2 .$$