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8. A string of linear density ρ is under tension T, and lies along the x-axis. Derive the wave equation for small transverse displacements $y(x,t)$ of the string. [6]

A finite string of length L lies between $x = a$ and $x = a + L$, and has its ends fixed with $y = 0$. Deduce forms of the initial displacement $y(x, t = 0)$ such that subsequently the displacement $y(x,t)$ retains the same shape, but has a different normalisation $f(t)$ i.e

$$y(x,t) = f(t) \times y(x, t = 0)$$

Find the function $f(t)$ for each of these initial displacements. [7]

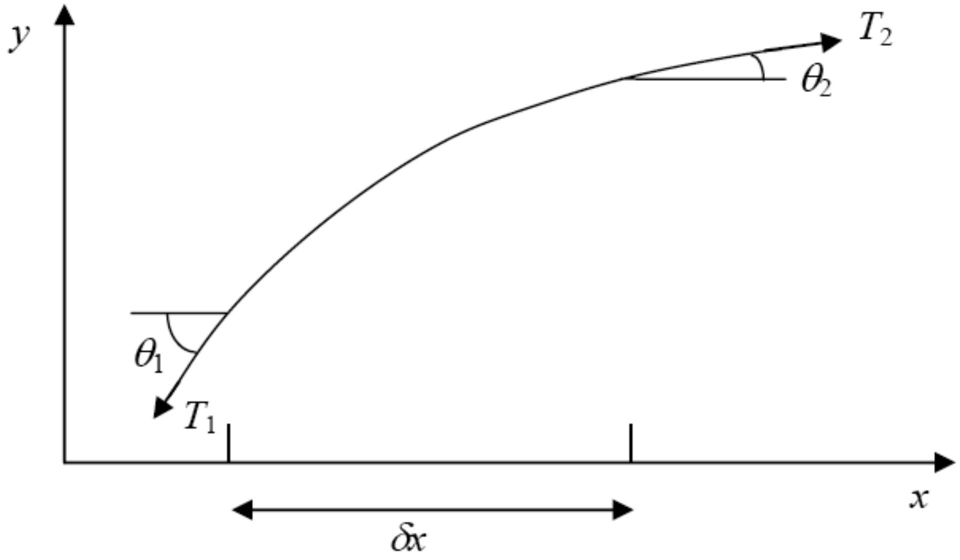
For such a string between $x = a$ and $x = a + L$, the initial displacement is

$$y(x, t = 0) = A \sin(2\pi(x - a)/L) \cos(\pi(x - a)/L)$$

Initially the string is at rest. Determine the subsequent displacement of the string. [7]

The Wave Equation

Transverse displacements of an elastic string of linear density (kg/m) ρ



Resolve horizontal forces : $T_1 \cos\theta_1 = T_2 \cos\theta_2$ for small θ, $\cos\theta \sim 1 \Rightarrow T_1 = T_2 = T$

Resolve vertical forces $T\sin\theta_2 - T\sin\theta_1 = (\rho\delta x)\dfrac{\partial^2 y}{\partial t^2}$

$$\therefore T\left[\left(\frac{\partial y}{\partial x}\right)_2 - \left(\frac{\partial y}{\partial x}\right)_1\right] = \rho\delta x\frac{\partial^2 y}{\partial t^2}$$

$$\left(\sin\theta \approx \tan\theta = \frac{\partial y}{\partial x}\right)$$

$$T\left[\left(\frac{\partial y}{\partial x}\right)_2 - \left(\frac{\partial y}{\partial x}\right)_1\right] = \rho\delta x\frac{\partial^2 y}{\partial t^2}$$

$$\left(\frac{\partial y}{\partial x}\right)_2 = \left(\frac{\partial y}{\partial x}\right)_1 + \frac{\partial}{\partial x}\left(\frac{\partial y}{\partial x}\right)\delta x + \dots \quad \Rightarrow \quad T\left(\frac{\partial^2 y}{\partial x^2}\right)\delta x = \rho\frac{\partial^2 y}{\partial t^2}\delta x$$

$$\Rightarrow \quad \frac{\partial^2 y}{\partial x^2} = \frac{\rho}{T}\frac{\partial^2 y}{\partial t^2}$$

This is a WAVE EQUATION with velocity $c = \sqrt{T / \rho}$
(hence larger tension or lighter string leads to faster waves)

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Let $x' = x - a$

Boundary conditions

$$y(x'=0,t) = y(x'=l,t) = 0 \quad \Rightarrow \quad y(x',t=0) = \sum_n a_n \sin\left(\frac{n\pi x'}{l}\right)$$

d'Alembert: general solution to wave equation $y(x',t) = f(kx' - \omega t) + g(kx' + \omega t)$

Wavenumber (k) — Angular frequency (ω)

$i.e.$ continuation to $t \neq 0$ must be of the form:

$$\sin(kx') \to a\sin(kx' - \omega t) + (1-a)\sin(kx' + \omega t)$$

$$= a(\sin kx' \cos\omega t - \cos kx' \sin\omega t) + (1-a)(\sin kx' \cos\omega t + \cos kx' \sin\omega t)$$

But boundary conditions apply at all times which requires $a = (1-a)$ $\quad i.e.\ a = \frac{1}{2}$

$$\Rightarrow \quad \sin(kx') \quad \to \quad \sin(kx')\cos(\omega t)$$

$$v = \sqrt{\frac{T}{\rho}} = \frac{\omega}{k}$$

$$i.e.\ \omega = \sqrt{\frac{T}{\rho}}\, k$$

General solution:

$$y(x',t) = \sum_n a_n \sin\left(\frac{n\pi x'}{l}\right)\cos\left(\frac{n\pi}{l}\sqrt{\frac{T}{\rho}}\, t\right)$$

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$$y(x',t)=\sum_n a_n \sin\left(\frac{n\pi(x-a)}{l}\right)\cos\left(\frac{n\pi}{l}\sqrt{\frac{T}{\rho}}\,t\right)$$

using $x'=x-a$

The initial displacement given can be written in this form:

$$y(x,t=0)=A\sin\left(2\pi(x-a)/l\right)\cos\left(\pi(x-a)/l\right)$$

$$=\frac{1}{2}A\left[\sin\left(3\pi(x-a)/l\right)+\sin\left(\pi(x-a)/l\right)\right]$$

Hence

$$y(x,t)=\frac{1}{2}A\left[\sin\left(\frac{3\pi}{l}(x-a)\right)\cos\left(\frac{3\pi}{l}\sqrt{\frac{T}{\rho}}t\right)+\sin\left(\pi(x-a)/l\right)\cos\left(\frac{\pi}{l}\sqrt{\frac{T}{\rho}}t\right)\right]$$

6. Two waves of slightly different wave number k_1 and k_2 but equal amplitudes are travelling in the same direction in a dispersive medium, i.e. one for which the phase velocity $v = \omega/k$ depends on k. (Here ω is the angular frequency of the wave). Write down an expression for the displacement of each of the two waves, and determine their resultant sum. Draw a labelled sketch for $t = 0$ to show this resultant. [7]

Explain what is meant by the group velocity g of a wave packet. Use your above analysis to argue that

$$g = \frac{\partial \omega}{\partial k} \ .$$ [6]

In a particular medium, the phase velocity is given by

$$v = v_0 + \alpha(k - k_0)$$

where v_0, α and k_0 are constants. Find the difference between the group velocity and the phase velocity for a wave packet made of waves with wave numbers close to k_0. [7]

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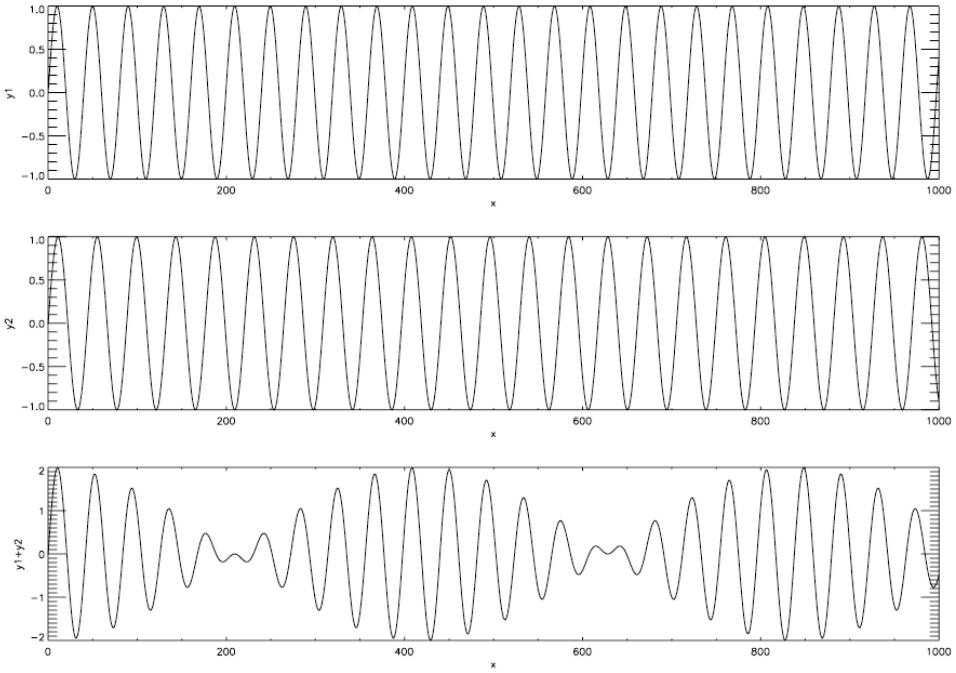
$$y_1 = A\sin\left[\left(k+\delta k\right)x-\left(\omega+\delta\omega\right)t\right]$$

$$y_2 = A\sin\left[\left(k-\delta k\right)x-\left(\omega-\delta\omega\right)t\right]$$

$$\delta k << k, \quad \delta\omega << \omega$$



$$y = y_1 + y_2 = 2A\cos\left(\delta k.x-\delta\omega.t\right)\sin\left(kx-\omega t\right)$$

Phase velocity $\mathrm{v} \simeq \dfrac{\omega}{\mathrm{k}}$

Group velocity $g = \dfrac{\delta\omega}{\delta k}$ (velocity of envelope)

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$$v_p = v_0 + \alpha(k - k_0) = \frac{\omega}{k} \quad \Rightarrow \quad \omega = k\left(v_0 + \alpha(k - k_0)\right)$$

$$v_g = \frac{\partial \omega}{\partial k} = \left(v_0 + \alpha(k - k_0)\right) + \alpha k = v_p + \alpha k$$

$$\boxed{v_g - v_p = \alpha k}$$

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6. Two identical long strings are attached to a point mass M. The strings are stretched along the x-axis and are under tension T. The equilibrium position of the mass is at the origin. The mass is now displaced slightly in the transverse direction y and subsequently released. Show that

$$M\left[\frac{\partial^2 y_2}{\partial t^2}\right]_{x=0} = T\left[\frac{\partial y_2}{\partial x} - \frac{\partial y_1}{\partial x}\right]_{x=0},$$

where y_1 and y_2 represent displacements of the string on either side of the mass. [6]

Show that the amplitude reflection and transmission coefficients for a wave incident on the mass are

$$r = \frac{-ip}{1+ip}$$

and

$$t = \frac{1}{1+ip},$$

where $p = \frac{1}{2}\frac{\omega^2 M}{Tk}$, k is the wavenumber $(\frac{2\pi}{\lambda})$ and ω the angular frequency. [10]

Sketch the variation of the phase change on reflection as a function of M for fixed ω and T. [4]

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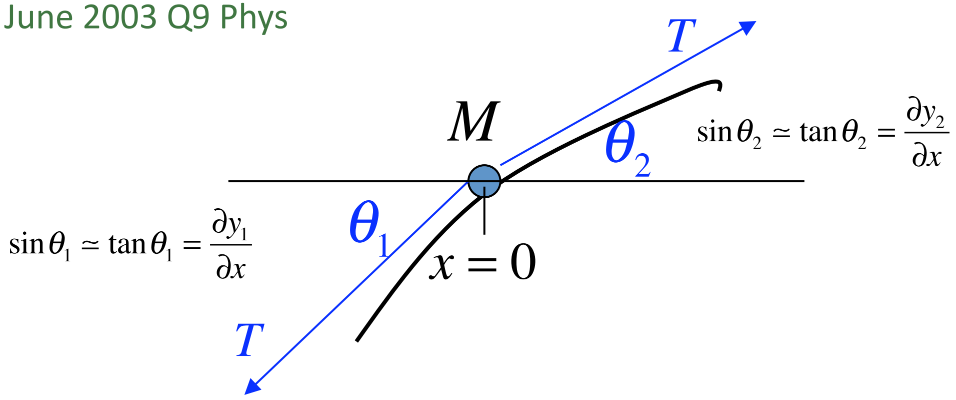

Newton's law

$$M \frac{\partial^2 y_2}{\partial t^2}\Big|_{x=0} = T\left[\frac{\partial y_2}{\partial x} - \frac{\partial y_1}{\partial x}\right]_{x=0}$$

$$y_{1,i} = e^{i(kx-\omega t)}$$

$$y_{1,r} = re^{-i(kx+\omega t)}$$

$$y_{2,t} = te^{i(kx-\omega t)}$$

$NB \quad T$ const $\Rightarrow \quad v=\sqrt{\frac{T}{\rho}} = \frac{\omega}{k}$ const

Hence k is constant since ω is constant

Boundary conditions at x=0:

$$y_{1,i} + y_{1,r} = y_{2,t} \quad \Rightarrow \quad 1 + r = t$$

$$M \frac{\partial^2 y_2}{\partial t^2}\Big|_{x=0} = T\left[\frac{\partial y_2}{\partial x} - \frac{\partial y_1}{\partial x}\right] \Rightarrow -M\omega^2 t = iTk\big(t-(1-r)\big) \Rightarrow 1-r = t(1-i\frac{M\omega^2}{Tk}) = t\big((1-2ip\big)$$

$$t = \frac{1}{1-ip}, \quad r = \frac{ip}{1-ip} \quad \text{where} \quad p = \frac{1}{2}\frac{\omega^2 M}{Tk}$$

Note: sign difference cf question

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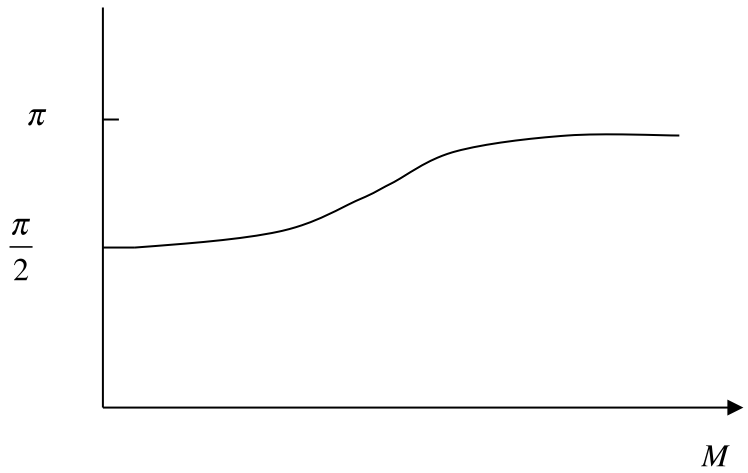
$$t = \frac{1}{1-ip}, \quad r = \frac{ip}{1-ip} \quad \text{where} \quad p = \frac{1}{2}\frac{\omega^2 M}{Tk}$$

$$r = \frac{ip}{1-ip} = \frac{ip}{1-ip}\frac{1+ip}{1+ip} = \frac{-p^2+ip}{1+p^2} = |r|e^{i\phi}$$

$$\tan\phi = -\frac{p}{p^2} = -\frac{2Tk}{\omega^2 M}, \quad \text{in 2nd quadrant since } \sin\phi \text{ is positive}$$



9. A long string lies along the x-axis and is under tension T. The displacement of the string from its equilibrium position at x is given by $y(x,t)$. By considering the forces acting on an element of the string show that y satisfies the wave equation. [5]

Two long strings of different densities ρ_1 and ρ_2 are joined together at $x = 0$. Write down the boundary conditions which must hold at $x = 0$ and use these to show that the power reflection and transmission coefficients R and T, respectively, for a wave incident on the boundary are

$$R = \left(\frac{\sqrt{\rho_1} - \sqrt{\rho_2}}{\sqrt{\rho_1} + \sqrt{\rho_2}}\right)^2,$$

$$T = \frac{4\sqrt{\rho_1}\sqrt{\rho_2}}{(\sqrt{\rho_1} + \sqrt{\rho_2})^2}\,.$$ [12]

What is the phase difference between the incident and reflected waves when ρ_1 is less than ρ_2? [3]

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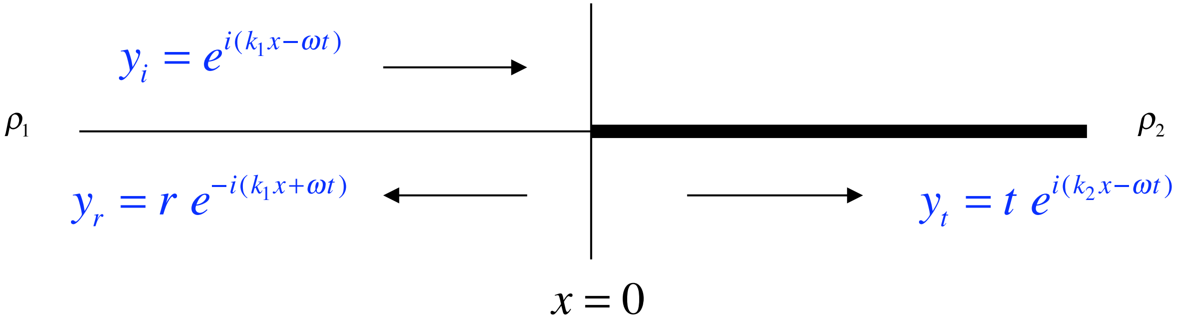

$$\mathrm{v}_{1,2} = \sqrt{\frac{T}{\rho_{1,2}}} = \frac{\omega}{k_{1,2}}$$

$$\Rightarrow \quad k_{1,2} = \omega \sqrt{\frac{\rho_{1,2}}{T}}$$

(tension constant in string)

Boundary conditions

$$y_1(x=0,t) = y_2(x=0,t) \qquad \text{(String continuous)}$$

$$\frac{\partial \mathrm{y}_1}{\partial \mathrm{t}}(x=0,t) = \frac{\partial \mathrm{y}_2}{\partial \mathrm{t}}(x=0,t) \qquad \text{(Forces continuous if no mass at join)}$$

$$1 + r = t$$
$$k_1(1-r) = k_2\, t$$

$$\Longrightarrow \quad r = \frac{1 - k_2/k_1}{1 + k_2/k_1} = \frac{1 - \sqrt{\rho_2/\rho_1}}{1 + \sqrt{\rho_2/\rho_1}}, \qquad t = \frac{2}{1 + k_2/k_1} = \frac{2}{1 + \sqrt{\rho_2/\rho_1}}$$

Phase difference between incident and reflected waves

$$r = \frac{\sqrt{\rho_1} - \sqrt{\rho_2}}{\sqrt{\rho_1} + \sqrt{\rho_2}} = |r| e^{i\phi},$$

r is negative for $\rho_1 < \rho_2$ *i.e.* $\phi = \pi$

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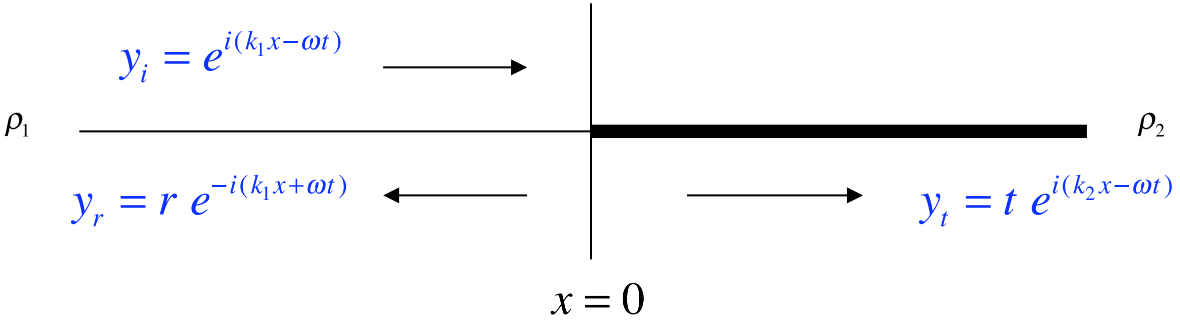

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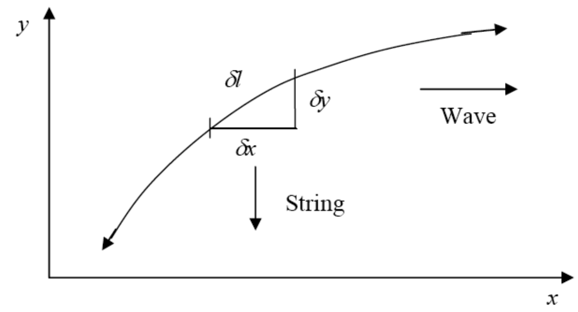
Power transmission

Energy of vibrating string



$$y = A\sin\left(kx - \omega t\right)$$

Kinetic Energy

$$KE \text{ of section} = \frac{1}{2}\rho\delta x\left(\frac{\partial y}{\partial t}\right)^2 = \frac{1}{2}\rho A^2\omega^2\cos^2\left(kx-\omega t\right)\delta x$$

$$KE = \frac{1}{2}\rho A^2\omega^2\int_x^{x+l\lambda}\cos^2\left(kx'-\omega t\right)dx' = \frac{1}{2}\rho A^2\omega^2\times\frac{l}{2}$$

$$KE/l = \frac{1}{4}\rho A^2\omega^2$$

Potential Energy

$$PE \text{ in stretched string element} = T\left(\delta l - \delta x\right) = T\delta x\left(\sqrt{1+\left(\frac{\delta y}{\delta x}\right)^2} - 1\right) \approx \frac{1}{2}TA^2k^2\cos^2\left(kx-\omega t\right)\delta x$$

$$PE = \frac{1}{2}TA^2k^2\int_x^{x+l\lambda}\cos^2\left(kx'-\omega t\right)dx'$$

$$PE/l = \frac{1}{4}TA^2k^2$$

$$\mathrm{v} = \omega/k = \sqrt{T/\rho} \quad\Rightarrow Tk^2 = \rho\omega^2 \quad\Rightarrow \text{PE=KE!}$$

(Example of virial theorem)

Energy flow

Total energy per wavelength, E/λ= KE+PE=$\frac{1}{2}\rho A^2\omega^2$

Distance travelled = vt

$$\text{Power=Energy flow/unit time} = \left(\frac{1}{2}\rho A^2\omega^2\right)\text{v}t/t$$

$$= \frac{1}{2}\rho\omega^2 A^2 \text{v}$$

$$= \frac{1}{2}\rho\omega^3 A^2 / k \qquad \text{v} = \frac{\omega}{k}$$

$$= \frac{1}{2}Tk^2 A^2 \text{v} \qquad Tk^2 = \rho\omega^2$$

$$= \frac{1}{2}T\omega k A^2$$

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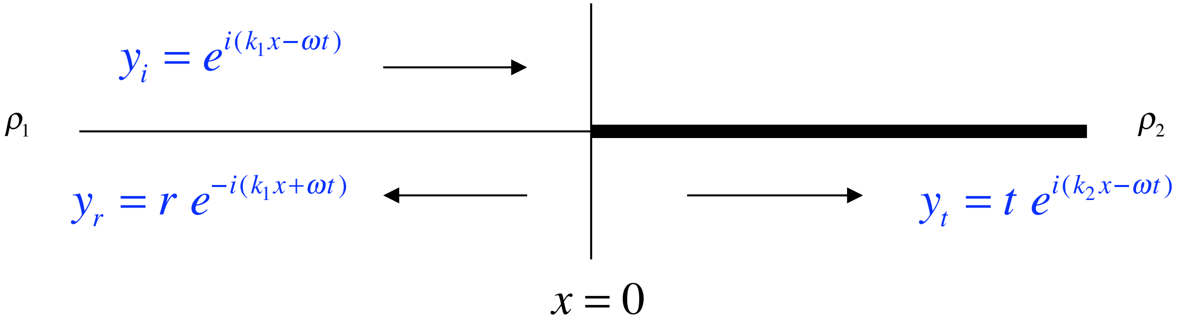

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Power transmission

$$P_{1,2} = \frac{1}{2} T \omega k_{1,2} |A|^2 \quad \Rightarrow$$

$$R = \frac{k_1}{k_1}|r|^2 = \left(\frac{\sqrt{\rho_1} - \sqrt{\rho_2}}{\sqrt{\rho_1} + \sqrt{\rho_2}}\right)^2 \qquad T = \frac{k_2}{k_1}|t|^2 = \sqrt{\frac{\rho_2}{\rho_1}}\frac{4}{\left(1 + \sqrt{\rho_2/\rho_1}\right)^2} = \frac{4\sqrt{\rho_1}\sqrt{\rho_2}}{\left(\sqrt{\rho_1} + \sqrt{\rho_2}\right)^2}$$