11. A mass attached to a spring executes driven, damped, simple harmonic motion. Explain the origin of each of the terms in the equation of motion

$$\ddot{x} + \gamma\dot{x} + (k/m)x = (F_0/m)e^{i\omega t}.$$ [3]

Show that a solution of this equation can be written as a complex number in the form $re^{i(\omega t-\phi)}$ where

$$r = \frac{F_0/m}{\left[\left(\omega_0^2 - \omega^2\right)^2 + \gamma^2\omega^2\right]^{\frac{1}{2}}},$$

$$\phi = \arctan\left\{\frac{\gamma\omega}{(\omega_0^2 - \omega^2)}\right\}$$

and $\omega_0^2 = k/m$. [6]

For a system subjected to a constant periodic force ($F(t) = F_0 \cos\omega t$) what is the average power transferred? Show that the maximum occurs at $\omega = \omega_0$ and that this has the value

$$P_{av} = \frac{F_0^2}{2m\gamma}.$$ [7]

Sketch a graph of the power against angular frequency and mark the half-power points at $\omega_1$and ω_2. For a sharp resonance show that $(\omega_1 - \omega_2) = \gamma$. [4]

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$$\ddot{x} + \gamma\dot{x} + \omega_0^2 x = \frac{F_0}{m} e^{i\omega t}$$

A solution is given by the Particular Integral

PI: Try $x = Pe^{i\omega t}$

$$P\left(-\omega^2 + i\gamma\omega + \omega_0^2\right) = \frac{F_0}{m}$$

$$x = \frac{F_0}{m}\left(-\omega^2 + i\gamma\omega + \omega_0^2\right)^{-1} e^{i\omega t} = re^{i(\omega t - \phi)}$$

$$r = \frac{F_0}{m} \frac{1}{\left(\left(\omega_0^2 - \omega^2\right)^2 + \gamma^2\omega^2\right)^{1/2}}, \qquad \phi = \tan^{-1}\left\{\frac{\gamma\omega}{\omega_0^2 - \omega^2}\right\}$$

For driving force $F_0 \cos\omega t = F_0 \operatorname{Re}\left(e^{i\omega t}\right)$ the solution is $\operatorname{Re}(x) = r\cos(\omega t - \phi)$

$$\text{Power in} = F\dot{x} = \left(F_0 \cos\omega t\right)\operatorname{Re}\left(\frac{F_0}{m}\frac{i\omega}{\left(\left(\omega_0^2 - \omega^2\right)^2 + \gamma^2\omega^2\right)^{1/2}} e^{i(\omega t - \phi)}\right)$$

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$$\text{Power in} = F\dot{x} = (F_0 \cos\omega t)\,\mathrm{Re}\left(\frac{F_0}{m}\frac{i\omega}{\left((\omega_0^2-\omega^2)^2+\gamma^2\omega^2\right)^{1/2}}e^{i(\omega t-\phi)}\right)$$

$$= -\left(\frac{F_0^2}{m}\frac{\omega}{\left((\omega_0^2-\omega^2)^2+\gamma^2\omega^2\right)^{1/2}}\right)\cos(\omega t)\sin(\omega t-\phi)$$

$$= -\frac{1}{2}\left(\frac{F_0^2}{m}\frac{\omega}{\left((\omega_0^2-\omega^2)^2+\gamma^2\omega^2\right)^{1/2}}\right)\left(\sin(2\omega t-\phi)+\sin(-\phi)\right)$$

$$\phi = \tan^{-1}\left\{\frac{\gamma\omega}{\omega_0^2-\omega^2}\right\}$$

$$\sin\phi = \frac{\gamma\omega}{\sqrt{(\omega_0^2-\omega^2)^2+(\gamma\omega)^2}}$$

$$\text{Average power} = \frac{1}{2}\left(\frac{F_0^2}{m}\frac{\omega}{\left((\omega_0^2-\omega^2)^2+\gamma^2\omega^2\right)^{1/2}}\right)\sin(\phi)$$

$$= \frac{1}{2}\left(\frac{F_0^2}{m}\frac{\gamma\omega^2}{\left((\omega_0^2-\omega^2)^2+\gamma^2\omega^2\right)}\right) = \frac{1}{2}\left(\frac{F_0^2}{m}\frac{\gamma}{\left((\omega_0^2/\omega^2-1)^2+\gamma^2\right)}\right)$$

Max at $\omega^2 = \omega_0^2$ with average power $P_{av}^{max} = \frac{F_0^2}{2m\gamma}$

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$$\phi = \tan^{-1}\left\{\frac{\gamma\omega}{\omega_0^2 - \omega^2}\right\}$$

$$\sin\phi = \frac{\gamma\omega}{\sqrt{\left(\omega_0^2 - \omega^2\right)^2 + (\gamma\omega)^2}}$$

$$\text{Average power} = \frac{1}{2}\left(\frac{F_0^2}{m}\frac{\omega}{\left(\left(\omega_0^2 - \omega^2\right)^2 + \gamma^2\omega^2\right)^{1/2}}\right)\sin(\phi)$$

$$= \frac{1}{2}\left(\frac{F_0^2}{m}\frac{\gamma\omega^2}{\left(\left(\omega_0^2 - \omega^2\right)^2 + \gamma^2\omega^2\right)}\right) = \frac{1}{2}\left(\frac{F_0^2}{m}\frac{\gamma}{\left(\left(\omega_0^2/\omega^2 - 1\right)^2 + \gamma^2\right)}\right)$$

Max at $\omega^2 = \omega_0^2$ with average power $P_{av}^{max} = \dfrac{F_0^2}{2m\gamma}$

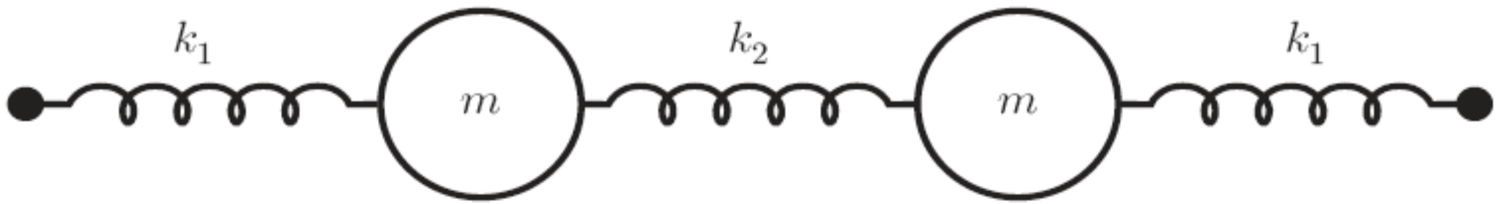
Half power points at $\left(\omega_0^2/\omega_{1,2}^2 - 1\right) = \pm\gamma$

$$i.e.\quad \omega_0^2/\omega_{1,2}^2 = 1 \pm \gamma \quad\Rightarrow\quad \omega_0/\omega_{1,2} = \sqrt{1\pm\gamma} \simeq 1 \pm \gamma/2$$

small γ

$$\frac{\omega_{1,2}}{\omega_0} = \frac{1}{1\pm\gamma/2} \simeq 1 \mp \frac{\gamma}{2} \quad\Rightarrow\quad \omega_2 - \omega_1 \simeq \gamma\omega$$

10. A linear mechanical system is constrained to move along a straight line. It consists of two identical masses m and three springs lying on a smooth table, as shown.



The end springs are fastened to fixed supports. The force constant of the two end springs is k_1 with an equilibrium extension s_1, while that of the middle spring connecting the masses is k_2 with an equilibrium extension s_2. Show that, when the displacements of the two masses from equilibrium are x_1 and x_2 respectively, the equations of motion are

$$\ddot{x}_1 + \frac{(k_1 + k_2)}{m} x_1 - \frac{k_2}{m} x_2 = 0,$$

$$\ddot{x}_2 + \frac{(k_1 + k_2)}{m} x_2 - \frac{k_2}{m} x_1 = 0.$$

[6]

Hence determine the frequencies of the two normal modes of the system and show for the case when $k_1 = k_2 = k$ that the two frequencies are $\omega_1 = \sqrt{k/m}$ and $\omega_2 = \sqrt{3k/m}$. [10]

What do the two cases $k_1 \gg k_2$ and $k_2 \gg k_1$ represent physically? [4]

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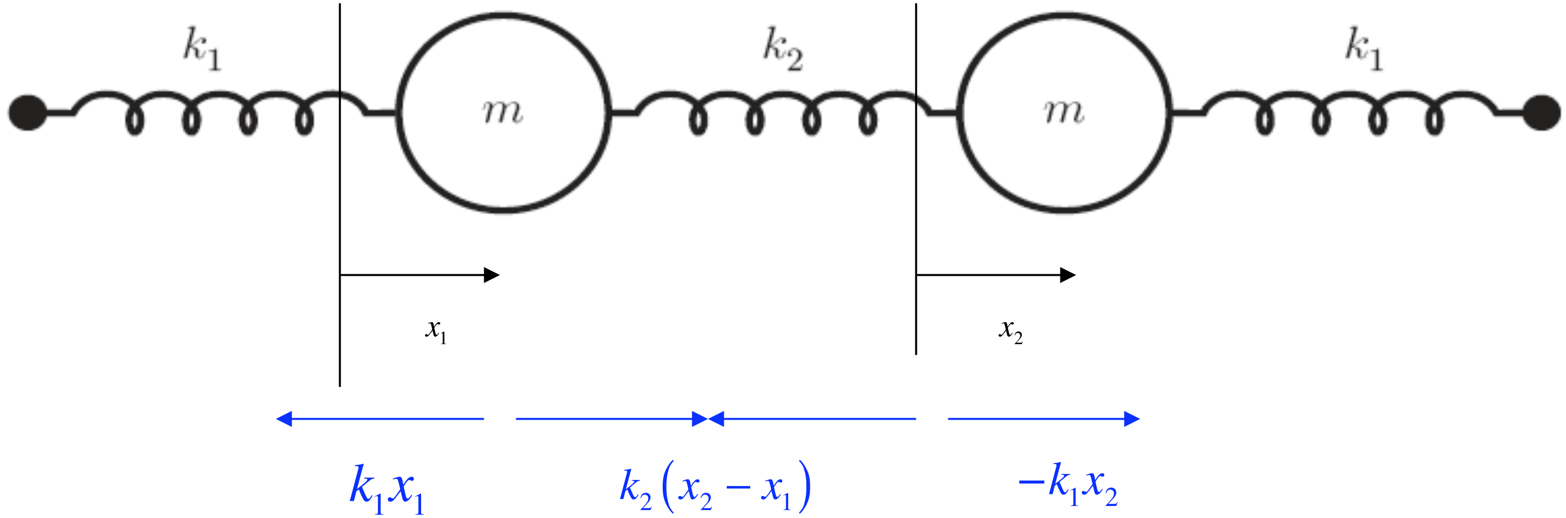

Newton's law for each mass gives:

$$\ddot{x}_1 + \frac{(k_1+k_2)}{m}x_1 - \frac{k_2}{m}x_2 = 0,$$

$$\ddot{x}_2 + \frac{(k_1+k_2)}{m}x_2 - \frac{k_2}{m}x_1 = 0.$$

$$\begin{pmatrix} \frac{d^2}{dt^2} + \frac{(k_1+k_2)}{m} & -\frac{k_2}{m} \\ -\frac{k_2}{m} & \frac{d^2}{dt^2} + \frac{(k_1+k_2)}{m} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Try $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} X \\ Y \end{pmatrix} e^{i\omega t}$

$$\begin{pmatrix} -\omega^2 + \frac{(k_1+k_2)}{m} & -\frac{k_2}{m} \\ -\frac{k_2}{m} & -\omega^2 + \frac{(k_1+k_2)}{m} \end{pmatrix} \begin{pmatrix} X \\ Y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

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Try $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} X \\ Y \end{pmatrix} e^{i\omega t}$

$$\begin{pmatrix} -\omega^2 + \frac{(k_1+k_2)}{m} & -\frac{k_2}{m} \\ -\frac{k_2}{m} & -\omega^2 + \frac{(k_1+k_2)}{m} \end{pmatrix} \begin{pmatrix} X \\ Y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Eigenvalues

$$Det \begin{pmatrix} -\omega^2 + \frac{(k_1+k_2)}{m} & -\frac{k_2}{m} \\ -\frac{k_2}{m} & -\omega^2 + \frac{(k_1+k_2)}{m} \end{pmatrix} = 0 \quad \Rightarrow \quad \omega_{1,2}^2 = \frac{(k_1+k_2)}{m} \pm \frac{k_2}{m}$$

$$k_1 = k_2 = k \quad \Rightarrow \quad \omega_1 = \sqrt{k/m}, \quad \omega_2 = \sqrt{3k/m}$$

$k_1 \gg k_2$ 2 nearly decoupled oscillators with frequency $\sqrt{\mathrm{k}_1 / m}$

$k_2 \gg k_1$ 1 nearly decoupled oscillator with frequency $\sqrt{2\mathrm{k}_2 / m} + CM$ motion

10. A double pendulum consists of light, inextensible strings, AB and BC, each of length ℓ. It is fixed at one end A and carries two particles, each of mass m, which hang under gravity. One of the particles is attached at the mid-point B while the other is located at C. The pendulum is constrained to move in a vertical plane. The angle between the vertical and AB is θ while the angle between BC and the vertical is ϕ. Show that, for small angles about the equilibrium position,

$$\ddot{\theta} + \frac{g}{\ell}(2\theta - \phi) = 0,$$

$$\ddot{\phi} + 2\frac{g}{\ell}(\phi - \theta) = 0.$$ [7]

Determine the normal frequencies for small oscillations of this system and show that the higher frequency is $\left(\sqrt{2}+1\right)$ times the lower frequency. Show also that, in both normal modes, the amplitude of ϕ is $\sqrt{2}$ times that of θ. [8]

Draw a sketch to show the instantaneous positions of the two masses at maximum amplitude for both the high and low frequency modes and calculate the difference in the frequency of the two modes for $\ell = 10\,\text{cm}$. [5]

[Take the acceleration due to gravity to be $g = 9{\cdot}8\,\text{ms}^{-2}$.]

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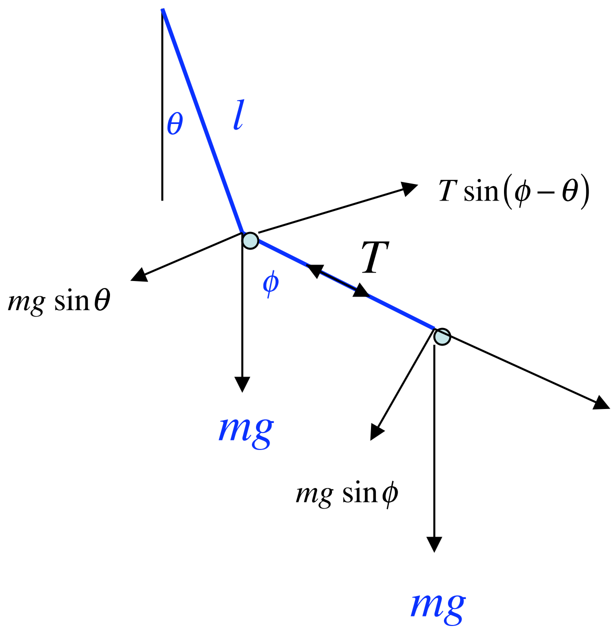

$$T = mg\cos\phi \simeq mg$$

$$ml\ddot{\theta} = -mg\sin\theta + mg\sin(\phi-\theta) \simeq -mg(2\theta-\phi)$$

$$\ddot{\theta} + \frac{g}{l}(2\theta-\phi) = 0$$

$$m\left(l\ddot{\theta} + l\ddot{\phi}\right) = -mg\sin\phi \quad \Rightarrow \quad \ddot{\phi} = -\frac{g}{l}\phi + \frac{g}{l}(2\theta-\phi)$$

$$\ddot{\phi} + \frac{2g}{l}(\phi-\theta) = 0$$

$$\begin{pmatrix} \frac{d^2}{dt^2} + \frac{2g}{l} & -\frac{g}{l} \\ -\frac{2g}{l} & \frac{d^2}{dt^2} + \frac{2g}{l} \end{pmatrix} \begin{pmatrix} \theta \\ \phi \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

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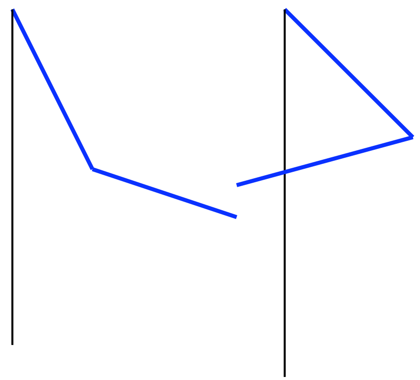
$$\begin{pmatrix} \frac{d^2}{dt^2}+\frac{2g}{l} & -\frac{g}{l} \\ -\frac{2g}{l} & \frac{d^2}{dt^2}+\frac{2g}{l} \end{pmatrix}\begin{pmatrix}\theta\\ \phi\end{pmatrix}=\begin{pmatrix}0\\0\end{pmatrix}$$

Try $\begin{pmatrix}\theta\\ \phi\end{pmatrix}=\begin{pmatrix}X\\Y\end{pmatrix}e^{i\omega t}$ $\quad \begin{pmatrix} -\omega^2+\frac{2g}{l} & -\frac{g}{l} \\ -\frac{2g}{l} & -\omega^2+\frac{2g}{l} \end{pmatrix}\begin{pmatrix}X\\Y\end{pmatrix}=\begin{pmatrix}0\\0\end{pmatrix}$

Eigenvalues

$$Det\begin{pmatrix} -\omega^2+\frac{2g}{l} & -\frac{g}{l} \\ -\frac{2g}{l} & -\omega^2+\frac{2g}{l} \end{pmatrix}=0 \quad\Rightarrow\quad \omega_{1,2}^2=\frac{g}{l}\left(2\pm\sqrt{2}\right)$$

$$\frac{\omega_1^2}{\omega_2^2}=\frac{2+\sqrt{2}}{2-\sqrt{2}}=\frac{2+\sqrt{2}}{2-\sqrt{2}}.\frac{2+\sqrt{2}}{2+\sqrt{2}}=\frac{\left(2+\sqrt{2}\right)^2}{2} \quad\Rightarrow\quad \frac{\omega_1}{\omega_2}=1+\sqrt{2}$$



Eigenvectors

$$\begin{pmatrix} -\frac{g}{l}\left(2\pm\sqrt{2}\right)+\frac{2g}{l} & -\frac{g}{l} \\ -\frac{2g}{l} & -\frac{g}{l}\left(2\pm\sqrt{2}\right)+\frac{2g}{l} \end{pmatrix}\begin{pmatrix}X\\Y\end{pmatrix}=\begin{pmatrix}0\\0\end{pmatrix} \quad\Rightarrow\quad \begin{pmatrix} \mp\sqrt{2} & -1 \\ -2 & \mp\sqrt{2} \end{pmatrix}\begin{pmatrix}X\\Y\end{pmatrix}=\begin{pmatrix}0\\0\end{pmatrix}$$

$$\frac{Y}{X}=\mp\sqrt{2}$$