Ordinary Differential Equations II

1. Find the general solution of

$$\frac{d^2y}{dx^2} - 4\frac{dy}{dx} + 3y = 10\cos x.$$

2. Show that the general solution of

$$\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + y = 2e^{-x} + x^3 + 2\cos x.$$

is $y = (A + Bx + x^2)e^{-x} + \sin x + x^3 - 6x^2 + 18x - 24$, where A, B are arbitrary constants.

3. Find the solution of

$$\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + y = 3e^x$$

for which $y = 3$ and $\frac{dy}{dx} = 0$ at $x = 0$.

4. Solve the differential equation

$$\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + (\beta^2 + 1)y = e^x \sin^2 x$$

for general values of the real parameter β. Explain why this solution fails for $\beta = 0$ and $\beta = 2$ and find solutions for these values of β.

5. A mass m is constrained to move in a straight line and is attached to a spring of strength $\lambda^2 m$ and a dashpot which produces a retarding force $-\alpha m v$, where v is the velocity of the mass. Find the steady state displacement of the mass when an amplitude-modulated periodic force $Am\cos pt \sin\omega t$ with $p \ll \omega$ and $\alpha \ll \omega$ is applied to it.

Show that for $\omega = \lambda$ the displacement of the amplitude-modulated wave is approximately given by

$$-A\frac{\cos\omega t \sin(pt + \phi)}{\sqrt{4\omega^2 p^2 + \alpha^2\omega^2}} \quad \text{where} \quad \cos\phi = \frac{2\omega p}{\sqrt{4\omega^2 p^2 + \alpha^2\omega^2}}.$$

where A is a constant.

6. When a varying couple $I\cos nt$ is applied to a torsional pendulum with natural period $2\pi/m$ and the moment of inertia I, the angle of the pendulum satisfies the equation of motion $\ddot{\theta} + m^2\theta = \cos nt$. The couple is first applied at $t = 0$ when the pendulum is at rest in equilibrium. Show that in the subsequent motion the root mean square angular displacement is $1/|m^2 - n^2|$ when the average is taken over a time large compared with $1/|m - n|$. Discuss the motion as $|m - n| \to 0$.

7. Verify that $y = x + 1$ is a solution of

$$(x^2 - 1)\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} + (x+1)\frac{\mathrm{d}y}{\mathrm{d}x} - y = 0.$$

Writing $y = (x+1)u$, show that $u' = \mathrm{d}u/\mathrm{d}x$ satisfies

$$\frac{\mathrm{d}u'}{\mathrm{d}x} + \frac{3x-1}{x^2-1}u' = 0.$$

Hence show that the general solution of the original equation is

$$y = K\left(\tfrac{1}{4}(x+1)\ln\frac{x-1}{x+1} + \tfrac{1}{2}\right) + K'(x+1),$$

where K and K' are arbitrary constants.

8. Find a continuous solution with continuous first derivative of the system

$$\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} + 2\frac{\mathrm{d}y}{\mathrm{d}x} + 2y = \sin x + f(x)$$

subject to $y(-\frac{1}{2}\pi) = y(\pi) = 0$, where

$$f(x) = \begin{cases} 0 & x \le 0 \\ x^2 & x > 0 \end{cases}.$$

[Hint: obtain a general solution for each of the cases $x < 0$ and $x > 0$ and then obtain relations between your four arbitrary constants by making the solutions agree at $x = 0$.]