Ordinary Differential Equations I

1. State the order of the following differential equations and whether they are linear or non-linear : (i) $\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} + k^2 y = f(x)$ (ii) $\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} + 2y\frac{\mathrm{d}y}{\mathrm{d}x} = \sin x$ (iii) $\frac{\mathrm{d}y}{\mathrm{d}x} + y^2 = yx$.

2. Solve the following differential equations using the method stated:

(a) **Separable** (i) $\frac{\mathrm{d}y}{\mathrm{d}x} = x\mathrm{e}^y/(1+x^2)$, $y = 0$ at $x = 0$. (ii) $\frac{\mathrm{d}x}{\mathrm{d}t} = (2tx^2 + t)/t^2 x - x)$

(b) **Almost separable** $\frac{\mathrm{d}y}{\mathrm{d}x} = 2(2x + y)^2$

(c) **Homogeneous** $2\frac{\mathrm{d}y}{\mathrm{d}x} = (xy + y^2)/x^2$

(d) **Homogeneous but for constants** $\frac{\mathrm{d}y}{\mathrm{d}x} = (x + y - 1)/(x - y - 2)$

(e) **Integrating Factor** (i) $\frac{\mathrm{d}y}{\mathrm{d}x} + y/x = 3$, $x = 0$ at $y = 0$. (ii) $\frac{\mathrm{d}x}{\mathrm{d}t} + x\cos t = \sin 2t$

(f) **Bernoulli** $\frac{\mathrm{d}y}{\mathrm{d}x} + y = xy^{2/3}$.

3. Solve the following first order differential equations :

(i) $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{x - y\cos x}{\sin x}$

(ii) $(3x + x^2)\frac{\mathrm{d}y}{\mathrm{d}x} = 5y - 8$

(iii) $\frac{\mathrm{d}y}{\mathrm{d}x} + \frac{2x}{y} = 3$

(iv) $\frac{\mathrm{d}y}{\mathrm{d}x} + y/x = 2x^{3/2}y^{1/2}$

(v) $2\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{y}{x} + \frac{y^3}{x^3}$

(vi) $xy\frac{\mathrm{d}y}{\mathrm{d}x} - y^2 = (x + y)^2\mathrm{e}^{-y/x}$

(vii) $x(x - 1)\frac{\mathrm{d}y}{\mathrm{d}x} + y = x(x - 1)^2$

(viii) $2x\frac{\mathrm{d}y}{\mathrm{d}x} - y = x^2$

(ix) $\frac{\mathrm{d}x}{\mathrm{d}t} = \cos(x + t)$, $x = \pi/2$ at $t = 0$

(x) $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{x - y}{x - y + 1}$

(xi) $\frac{\mathrm{d}x}{\mathrm{d}y} = \cos 2y - x\cot y$, $x = 1/2$ at $y = \pi/2$

4. L_1 is the differential operator

$$L_1 = \left(\frac{\mathrm{d}}{\mathrm{d}x} + 2\right).$$

Evaluate (i) $L_1 x^2$, (ii) $L_1\big(x\mathrm{e}^{2x}\big)$, (iii) $L_1\big(x\mathrm{e}^{-2x}\big)$.

5. L_2 is the differential operator

$$L_2 = \left(\frac{\mathrm{d}}{\mathrm{d}x} - 1\right).$$

Express the operator $L_3 = L_2 L_1$ in terms of $\frac{\mathrm{d}^2}{\mathrm{d}x^2}, \frac{\mathrm{d}}{\mathrm{d}x}$, etc. Show that $L_1 L_2 = L_2 L_1$.

6. By introducing a new variable $Y = (4y - x)$, or otherwise, show that the solution of the o.d.e.

$$\frac{\mathrm{d}y}{\mathrm{d}x} - 16y^2 + 8xy = x^2$$

satisfies $4y - x - \frac{1}{2} = A(4y - x + \frac{1}{2})\mathrm{e}^{4x}$, where A is an arbitrary constant.

7. Solve the o.d.e.

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{(3x^2 + 2xy + y^2)\sin x - (6x + 2y)\cos x}{(2x + 2y)\cos x}.$$

[Hint: look for a function $f(x, y)$ whose differential $\mathrm{d}f$ gives the o.d.e.]

8. The equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} + ky = y^n \sin x,$$

where k and n are constants, is linear and homogeneous for $n = 1$. State a property of the solutions to this equation for $n = 1$ that is **not** true for $n \neq 1$.

Solve the equation for $n \neq 1$ by making the substitution $z = y^{1-n}$.