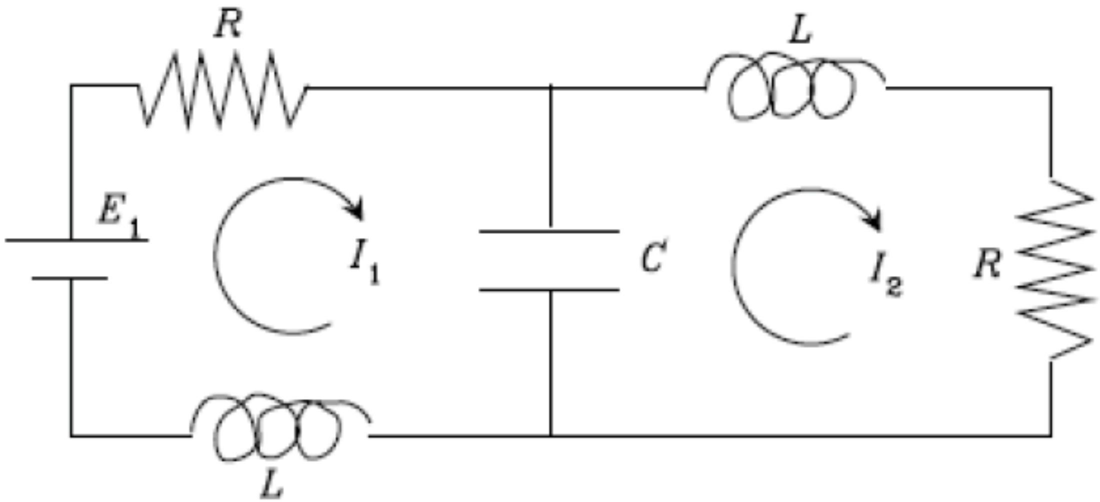
Ex 2

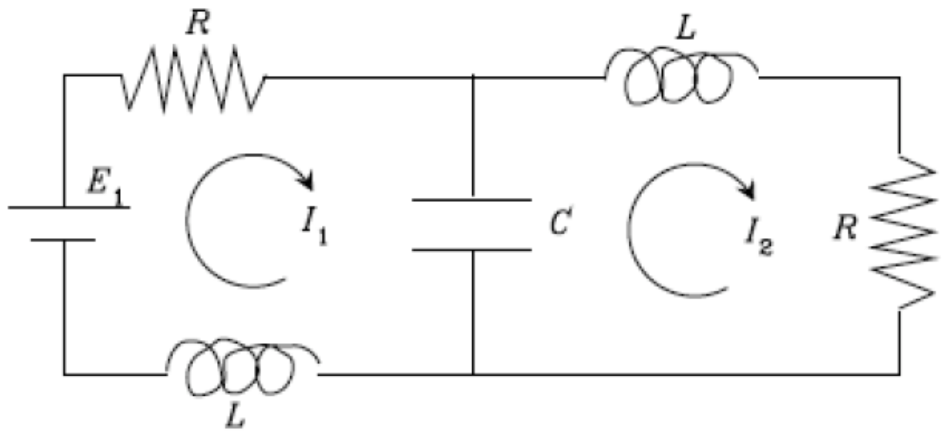
LCR circuits

The dynamics of a linear electrical circuit is governed by a system of linear equations with constant coefficients. These may be solved by the general matrix technique. In many cases they may be more easily solved by judicious addition and subtraction.

Ex 1



$$R^2 C < 8L$$



Using Kirchoff's laws :

$$RI_1 + \frac{Q}{C} + L\frac{\mathrm{d}I_1}{\mathrm{d}t} = E_1$$

$$L\frac{\mathrm{d}I_2}{\mathrm{d}t} + RI_2 - \frac{Q}{C} = 0.$$

Differentiate to eliminate Q

$$\frac{\mathrm{d}^2 I_1}{\mathrm{d}t^2} + \frac{R}{L}\frac{\mathrm{d}I_1}{\mathrm{d}t} + \frac{1}{LC}(I_1 - I_2) = \frac{\mathrm{d}E_1}{\mathrm{d}t} = 0$$

$$\frac{\mathrm{d}^2 I_2}{\mathrm{d}t^2} + \frac{R}{L}\frac{\mathrm{d}I_2}{\mathrm{d}t} - \frac{1}{LC}(I_1 - I_2) = 0.$$

$$\frac{d^2 I_1}{dt^2} + \frac{R}{L}\frac{dI_1}{dt} + \frac{1}{LC}(I_1 - I_2) = 0$$

$$\frac{d^2 I_2}{dt^2} + \frac{R}{L}\frac{dI_2}{dt} - \frac{1}{LC}(I_1 - I_2) = 0$$

$$\Rightarrow \begin{pmatrix} \frac{d^2}{dt^2} + \frac{R}{L}\frac{d}{dt} + \frac{1}{LC} & -\frac{1}{LC} \\ -\frac{1}{LC} & \frac{d^2}{dt^2} + \frac{R}{L}\frac{d}{dt} + \frac{1}{LC} \end{pmatrix} \begin{pmatrix} I_1 \\ I_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Try $\begin{pmatrix} I_1 \\ I_2 \end{pmatrix} = \begin{pmatrix} X \\ Y \end{pmatrix} e^{\alpha t}$

$$\Rightarrow \begin{pmatrix} \alpha^2 + \frac{R}{L}\alpha + \frac{1}{LC} & -\frac{1}{LC} \\ -\frac{1}{LC} & \alpha^2 + \frac{R}{L}\alpha + \frac{1}{LC} \end{pmatrix} \begin{pmatrix} X \\ Y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Auxiliary equations :

$$\alpha^2 + \frac{R}{L}\alpha = 0 \qquad \alpha = 0, \quad -R/L$$

Eigenvalues

$$\alpha^2 + \frac{R}{L}\alpha + \frac{2}{LC} = 0 \quad \Rightarrow \quad \alpha = -\tfrac{1}{2}\frac{R}{L} \pm \frac{i}{\sqrt{LC}}\sqrt{2 - \tfrac{1}{4}CR^2/L} = -\tfrac{1}{2}\frac{R}{L} \pm i\omega_R.$$

Eigenvalues

$$\frac{d^2 I_1}{dt^2}+\frac{R}{L}\frac{dI_1}{dt}+\frac{1}{LC}(I_1-I_2)=0$$
$$\frac{d^2 I_2}{dt^2}+\frac{R}{L}\frac{dI_2}{dt}-\frac{1}{LC}(I_1-I_2)=0$$

$$\Rightarrow \begin{pmatrix} \frac{d^2}{dt^2}+\frac{R}{L}\frac{d}{dt}+\frac{1}{LC} & -\frac{1}{LC} \\ -\frac{1}{LC} & \frac{d^2}{dt^2}+\frac{R}{L}\frac{d}{dt}+\frac{1}{LC} \end{pmatrix}\begin{pmatrix} I_1 \\ I_2 \end{pmatrix}=\begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Try $\begin{pmatrix} I_1 \\ I_2 \end{pmatrix}=\begin{pmatrix} X \\ Y \end{pmatrix}e^{\alpha t}$

$$\Rightarrow \begin{pmatrix} \alpha^2+\frac{R}{L}\alpha+\frac{1}{LC} & -\frac{1}{LC} \\ -\frac{1}{LC} & \alpha^2+\frac{R}{L}\alpha+\frac{1}{LC} \end{pmatrix}\begin{pmatrix} X \\ Y \end{pmatrix}=\begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Eigenvector equations :

$$\alpha^2+\frac{R}{L}\alpha=0$$

$\alpha=0, \quad -R/L$ Eigenvalues

$$\begin{pmatrix} +\frac{1}{LC} & -\frac{1}{LC} \\ -\frac{1}{LC} & +\frac{1}{LC} \end{pmatrix}\begin{pmatrix} X \\ Y \end{pmatrix}=\begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$\begin{pmatrix} X \\ Y \end{pmatrix}=Ae^{i\phi}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ Eigenvectors

$$\begin{pmatrix} I_1 \\ I_2 \end{pmatrix}=\left(A_0+A_1e^{-\frac{R}{L}t}\right)\begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$\frac{d^2 I_1}{dt^2} + \frac{R}{L}\frac{dI_1}{dt} + \frac{1}{LC}(I_1 - I_2) = 0$$

$$\frac{d^2 I_2}{dt^2} + \frac{R}{L}\frac{dI_2}{dt} - \frac{1}{LC}(I_1 - I_2) = 0$$

$$\Rightarrow \begin{pmatrix} \frac{d^2}{dt^2} + \frac{R}{L}\frac{d}{dt} + \frac{1}{LC} & -\frac{1}{LC} \\ -\frac{1}{LC} & \frac{d^2}{dt^2} + \frac{R}{L}\frac{d}{dt} + \frac{1}{LC} \end{pmatrix} \begin{pmatrix} I_1 \\ I_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Try $\begin{pmatrix} I_1 \\ I_2 \end{pmatrix} = \begin{pmatrix} X \\ Y \end{pmatrix} e^{\alpha t}$

$$\Rightarrow \begin{pmatrix} \alpha^2 + \frac{R}{L}\alpha + \frac{1}{LC} & -\frac{1}{LC} \\ -\frac{1}{LC} & \alpha^2 + \frac{R}{L}\alpha + \frac{1}{LC} \end{pmatrix} \begin{pmatrix} X \\ Y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Eigenvector equations :

$$\alpha^2 + \frac{R}{L}\alpha + \frac{2}{LC} = 0 \quad \Rightarrow \quad \alpha = -\tfrac{1}{2}\frac{R}{L} \pm \frac{i}{\sqrt{LC}}\sqrt{2 - \tfrac{1}{4}CR^2/L} = -\tfrac{1}{2}\frac{R}{L} \pm i\omega_R.$$

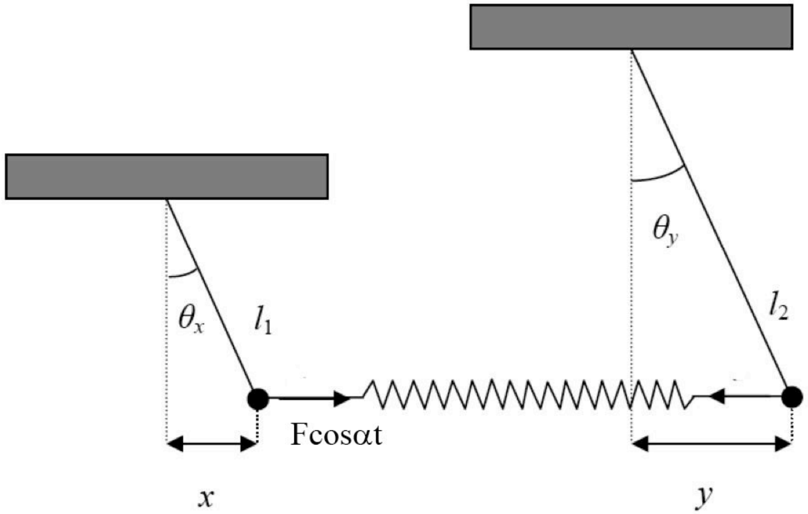
$$\begin{pmatrix} -\frac{1}{LC} & -\frac{1}{LC} \\ -\frac{1}{LC} & -\frac{1}{LC} \end{pmatrix} \begin{pmatrix} X \\ Y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} X \\ Y \end{pmatrix} = Be^{i\phi} \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

Eigenvectors

$$\begin{pmatrix} I_1 \\ I_2 \end{pmatrix} = \mathrm{Re}\left\{ \begin{pmatrix} X \\ Y \end{pmatrix} e^{\alpha t} \right\} = Be^{-\frac{R}{2L}} \cos(\omega_R t + \phi) \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

The damped driven pendulum



$$m_1\ddot{x} = -\gamma\dot{x} - m_1 g x / l_1 + k\left(y - x\right) + F\cos\alpha t$$

$$m_2\ddot{y} = -\gamma\dot{y} - m_2 g y / l_2 - k\left(y - x\right)$$

$$\begin{pmatrix} \frac{d^2}{dt^2} + \frac{\gamma}{m_1}\frac{d}{dt} + \left(\frac{g}{l_1} + \frac{k}{m_1}\right) & -\frac{k}{m_1} \\ -\frac{k}{m_2} & \frac{d^2}{dt^2} + \frac{\gamma}{m_2}\frac{d}{dt} + \left(\frac{g}{l_2} + \frac{k}{m_2}\right) \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \frac{F}{m_1}\begin{pmatrix} 1 \\ 0 \end{pmatrix}\mathrm{Re}\left(e^{i\alpha t}\right)$$

$$\begin{pmatrix} \frac{d^2}{dt^2}+\frac{\gamma}{m_1}\frac{d}{dt}+\left(\frac{g}{l_1}+\frac{k}{m_1}\right) & -\frac{k}{m_1} \\ -\frac{k}{m_2} & \frac{d^2}{dt^2}+\frac{\gamma}{m_2}\frac{d}{dt}+\left(\frac{g}{l_2}+\frac{k}{m_2}\right) \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \frac{F}{m_1}\begin{pmatrix} 1 \\ 0 \end{pmatrix}\mathrm{Re}\left(e^{i\alpha t}\right)$$

CF

$$\begin{pmatrix} x \\ y \end{pmatrix} = \mathrm{Re}\left(\begin{pmatrix} X \\ Y \end{pmatrix} e^{i\omega t}\right)$$

$$\begin{pmatrix} -\omega^2+i\frac{\gamma}{m_1}\omega+\left(\frac{g}{l_1}+\frac{k}{m_1}\right) & -\frac{k}{m_1} \\ -\frac{k}{m_2} & -\omega^2+i\frac{\gamma}{m_2}\omega+\left(\frac{g}{l_2}+\frac{k}{m_2}\right) \end{pmatrix}\begin{pmatrix} X \\ Y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \qquad (1)$$

$$\begin{vmatrix} -\omega^2+i\frac{\gamma}{m_1}\omega+\left(\frac{g}{l_1}+\frac{k}{m_1}\right) & -\frac{k}{m_1} \\ -\frac{k}{m_2} & -\omega^2+i\frac{\gamma}{m_2}\omega+\left(\frac{g}{l_2}+\frac{k}{m_2}\right) \end{vmatrix} = 0$$

Eigenvalue eq.

Substitute (complex) eigenvalues in (1) to obtain eigenvectors

Simple case $m_1 = m_2 = m \quad l_1 = l_2 = l$

$$\begin{vmatrix} -\omega^2 - i\frac{\gamma}{m}\omega + \left(\frac{g}{l} + \frac{k}{m}\right) & -\frac{k}{m} \\ -\frac{k}{m} & -\omega^2 - i\frac{\gamma}{m}\omega + \left(\frac{g}{l} + \frac{k}{m}\right) \end{vmatrix} = 0$$

$$-\omega^2 - i\frac{\gamma}{m}\omega + \left(\frac{g}{l}\right) = 0 \qquad \text{or} \qquad -\omega^2 - i\frac{\gamma}{m}\omega + \left(\frac{g}{l} + \frac{2k}{m}\right) = 0$$

$$\bar{\omega}_{1,2} = i\frac{\gamma}{2m} \pm \sqrt{\omega_{1,2}^2 - \left(\frac{\gamma}{2m}\right)^2}$$ Eigenvalues

$$\omega_1^2 = \frac{g}{l} \quad \text{or} \quad \omega_2^2 = \frac{g}{l} + 2\frac{k}{m}$$

$\gamma = 0$ eigenvalues (c.f. previous result)

Eigenvectors

$$\begin{pmatrix} -\omega^2 - i\frac{\gamma}{m}\omega + \left(\frac{g}{l}+\frac{k}{m}\right) & -\frac{k}{m} \\ -\frac{k}{m} & -\omega^2 - i\frac{\gamma}{m}\omega + \left(\frac{g}{l}+\frac{k}{m}\right) \end{pmatrix}\begin{pmatrix} X \\ Y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$-\omega^2 - i\frac{\gamma}{m}\omega + \left(\frac{g}{l}\right) = 0 \qquad \text{or} \qquad -\omega^2 - i\frac{\gamma}{m}\omega + \left(\frac{g}{l}+\frac{2k}{m}\right) = 0$$

$$\overline{\omega}_{1,2} = i\frac{\gamma}{2m} \pm \sqrt{\omega_{1,2}^2 - \left(\frac{\gamma}{2m}\right)^2}$$

Eigenvalues

$\omega = \overline{\omega}_1$

$$\begin{pmatrix} \frac{k}{m} & -\frac{k}{m} \\ -\frac{k}{m} & \frac{k}{m} \end{pmatrix}\begin{pmatrix} X_1 \\ Y_1 \end{pmatrix} = 0$$

$$\begin{pmatrix} X_1 \\ Y_1 \end{pmatrix} = A_1 e^{i\phi_1}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$\omega = \overline{\omega}_2$

$$\begin{pmatrix} -\frac{k}{m} & -\frac{k}{m} \\ -\frac{k}{m} & -\frac{k}{m} \end{pmatrix}\begin{pmatrix} X_2 \\ Y_2 \end{pmatrix} = 0$$

$$\begin{pmatrix} X_2 \\ Y_2 \end{pmatrix} = A_2 e^{i\phi_2}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

Eigenvectors

Eigenvectors

$$\begin{pmatrix} -\omega^2 - i\frac{\gamma}{m}\omega + \left(\frac{g}{l}+\frac{k}{m}\right) & -\frac{k}{m} \\ -\frac{k}{m} & -\omega^2 - i\frac{\gamma}{m}\omega + \left(\frac{g}{l}+\frac{k}{m}\right) \end{pmatrix} \begin{pmatrix} X \\ Y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$-\omega^2 - i\frac{\gamma}{m}\omega + \left(\frac{g}{l}\right) = 0 \qquad \text{or} \qquad -\omega^2 - i\frac{\gamma}{m}\omega + \left(\frac{g}{l}+\frac{2k}{m}\right) = 0$$

$$\bar{\omega}_{1,2} = i\frac{\gamma}{2m} \pm \sqrt{\omega_{1,2}^2 - \left(\frac{\gamma}{2m}\right)^2}$$

Eigenvalues

$$\begin{pmatrix} x \\ y \end{pmatrix} = \mathrm{Re}\left(A_1 e^{i\phi_1} \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{-\frac{\gamma}{2m}t} e^{it\sqrt{\omega_1^2 - \left(\frac{\gamma}{2m}\right)^2}} \right) + \mathrm{Re}\left(A_2 e^{i\phi_2} \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{-\frac{\gamma}{2m}t} e^{it\sqrt{\omega_2^2 - \left(\frac{\gamma}{2m}\right)^2}} \right)$$

$$= e^{-\frac{\gamma}{2m}t}\left(A_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} \cos\left(t\sqrt{\omega_1^2 - \left(\frac{\gamma}{2m}\right)^2} + \phi_1 \right) + A_2 \begin{pmatrix} 1 \\ -1 \end{pmatrix} \cos\left(t\sqrt{\omega_2^2 - \left(\frac{\gamma}{2m}\right)^2} + \phi_2 \right) \right)$$

Decoupling method

$$m\ddot{x} = -mg\frac{x}{l} + k(y-x) - \gamma\dot{x}$$

$$m\ddot{y} = -mg\frac{y}{l} - k(y-x) - \gamma\dot{y}$$

$$m(\ddot{x}+\ddot{y}) = -m\frac{g}{l}(x+y) - \gamma(\dot{x}+\dot{y})$$

$$m(\ddot{x}-\ddot{y}) = (-m\frac{g}{l} - 2k)(x-y) - \gamma(\dot{x}-\dot{y})$$

$$\ddot{q}_1 + \frac{\gamma}{m}\dot{q}_1 + \omega_1^2 q_1 = 0$$

$$\ddot{q}_2 + \frac{\gamma}{m}\dot{q}_2 + \omega_2^2 q_2 = 0$$

Two 2nd order differential equations - techniques discussed last term give

$$q_1 = A_1 \cos(\omega_1' t + \phi_1)e^{-\alpha t} \qquad \omega_1' = \sqrt{\omega_1^2 - \alpha^2}$$

$$q_2 = A_2 \cos(\omega_2' t + \phi_2)e^{-\alpha t} \qquad \omega_2' = \sqrt{\omega_2^2 - \alpha^2}$$

$$\alpha = \frac{\gamma}{2m}$$