Systems of linear equations with constant coefficients and driving terms

Ex 1

$$\frac{\mathrm{d}x}{\mathrm{d}t}+\frac{\mathrm{d}y}{\mathrm{d}t}+y=t,$$
$$-\frac{\mathrm{d}y}{\mathrm{d}t}+3x+7y=\mathrm{e}^{2t}-1.$$

Known variables on RHS

$$L\begin{pmatrix}x\\y\end{pmatrix}\equiv\begin{pmatrix}\frac{\mathrm{d}}{\mathrm{d}t} & +\frac{\mathrm{d}}{\mathrm{d}t}+1\\ 3 & -\frac{\mathrm{d}}{\mathrm{d}t}+7\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix}=\begin{pmatrix}\frac{\mathrm{d}x}{\mathrm{d}t} & +\frac{\mathrm{d}y}{\mathrm{d}t}+y\\ 3x & -\frac{\mathrm{d}y}{\mathrm{d}t}+7y\end{pmatrix}.$$

$$L\begin{pmatrix}x\\y\end{pmatrix}=\begin{pmatrix}t\\e^{2t}-1\end{pmatrix}$$

$$L\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} t \\ e^{2t} - 1 \end{pmatrix}$$

Complementary function :

$$L\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Set $\begin{pmatrix} x \\ y \end{pmatrix} = \mathrm{Re}\begin{pmatrix} X\mathrm{e}^{\alpha t} \\ Y\mathrm{e}^{\alpha t} \end{pmatrix}$ α, X, Y complex nos to be determined

(In this case we will find α is real so X,Y will be taken real too)

$$L\begin{pmatrix} X\mathrm{e}^{\alpha t} \\ Y\mathrm{e}^{\alpha t} \end{pmatrix} \equiv \begin{pmatrix} \dfrac{\mathrm{d}\mathrm{e}^{\alpha t}}{\mathrm{d}t} & +\dfrac{\mathrm{d}\mathrm{e}^{\alpha t}}{\mathrm{d}t} + \mathrm{e}^{\alpha t} \\ 3\mathrm{e}^{\alpha t} & -\dfrac{\mathrm{d}\mathrm{e}^{\alpha t}}{\mathrm{d}t} + 7\mathrm{e}^{\alpha t} \end{pmatrix} \begin{pmatrix} X \\ Y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

$$\Rightarrow \quad \begin{pmatrix} \alpha & \alpha + 1 \\ 3 & 7 - \alpha \end{pmatrix} \begin{pmatrix} X \\ Y \end{pmatrix} = 0$$

$$\begin{pmatrix} \alpha & \alpha+1 \\ 3 & 7-\alpha \end{pmatrix}\begin{pmatrix} X \\ Y \end{pmatrix} = 0$$

"Eigenvalue equation"

$$\Rightarrow \begin{vmatrix} \alpha & \alpha+1 \\ 3 & 7-\alpha \end{vmatrix} = 0$$

$$\Rightarrow \quad \alpha(7-\alpha) - 3(\alpha+1) = 0 \quad \Rightarrow \quad \alpha = 3, \alpha = 1$$

Eigenvalues

The allowed values of α determine the allowed values of X/Y :

$$\alpha = 3 \quad \Rightarrow \quad 3X + 4Y = 0 \quad \Rightarrow \quad Y = -\tfrac{3}{4}X$$

$$\alpha = 1 \quad \Rightarrow \quad X + 2Y = 0 \quad \Rightarrow \quad Y = -\tfrac{1}{2}X$$

"Eigenvectors"

$$(\alpha = 1 \quad \Rightarrow \quad 3X + 6Y = 0 \quad \Rightarrow \quad Y = -\tfrac{1}{2}X)$$

Real constants

$$\text{Hence CF is } \begin{pmatrix} x \\ y \end{pmatrix} = X_a \begin{pmatrix} 1 \\ -\frac{3}{4} \end{pmatrix} e^{3t} + X_b \begin{pmatrix} 1 \\ -\frac{1}{2} \end{pmatrix} e^{t}.$$

Particular Integral

$$L\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \dfrac{dx}{dt} & +\dfrac{dy}{dt} + y \\ 3x & -\dfrac{dy}{dt} + 7y \end{pmatrix} = \begin{pmatrix} t \\ e^{2t} - 1 \end{pmatrix}$$

To find PI we use trial functions

Polynomial part :

$$L\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} t \\ -1 \end{pmatrix}$$

$$\text{Try} \quad \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} X_0 + X_1 t \\ Y_0 + Y_1 t \end{pmatrix} \qquad = \begin{pmatrix} -\frac{28}{9} - \frac{7}{3}t \\ \frac{4}{3} + t \end{pmatrix}$$

$$\begin{pmatrix} X_1 + Y_1 + Y_1 t + Y_0 \\ 3(X_0 + X_1 t) - Y_1 + 7(Y_0 + Y_1 t) \end{pmatrix} = \begin{pmatrix} t \\ -1 \end{pmatrix}$$

$$X_1 = -\tfrac{7}{3}$$
$$Y_0 = -1 + \tfrac{7}{3} = \tfrac{4}{3}$$
$$X_0 = -\tfrac{7}{3} Y_0 = -\tfrac{28}{9}$$

$$Y_1 = 1 \qquad X_1 + Y_0 = -1$$

$$3X_0 + 7Y_0 = 0; \quad 3X_1 + 7Y_1 = 0$$

Particular Integral

$$L\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \frac{dx}{dt} + \frac{dy}{dt} + y \\ 3x - \frac{dy}{dt} + 7y \end{pmatrix} = \begin{pmatrix} t \\ e^{2t} - 1 \end{pmatrix}$$

Exponential part :

$$L\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ e^{2t} \end{pmatrix}$$

Try $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} X \\ Y \end{pmatrix} e^{2t}$ $= \begin{pmatrix} -3 \\ 2 \end{pmatrix} e^{2t}$

$$2X + (2+1)Y = 0 \quad \Rightarrow \quad X = -\tfrac{3}{2}Y$$
$$3X + (-2+7)Y = 1 \quad \Rightarrow \quad (-\tfrac{9}{2}+5)Y = 1$$

$$X = -3$$
$$Y = 2$$

The full solution :

$$\begin{pmatrix} x \\ y \end{pmatrix} = X_a \begin{pmatrix} 1 \\ -\frac{3}{4} \end{pmatrix} e^{3t} + X_b \begin{pmatrix} 1 \\ -\frac{1}{2} \end{pmatrix} e^{t} + \begin{pmatrix} -3 \\ 2 \end{pmatrix} e^{2t} + \begin{pmatrix} -\frac{28}{9} - \frac{7}{3}t \\ \frac{4}{3} + t \end{pmatrix}$$

Initial conditions

The full solution :

$$\begin{pmatrix} x \\ y \end{pmatrix} = X_a \begin{pmatrix} 1 \\ -\frac{3}{4} \end{pmatrix} e^{3t} + X_b \begin{pmatrix} 1 \\ -\frac{1}{2} \end{pmatrix} e^{t} + \begin{pmatrix} -3 \\ 2 \end{pmatrix} e^{2t} + \begin{pmatrix} -\frac{28}{9} - \frac{7}{3} t \\ \frac{4}{3} + t \end{pmatrix}$$

The constants must be fixed by initial conditions e.g.

$$\begin{cases} \dot{x}(0) = -\frac{19}{3} \\ \dot{y}(0) = 3 \end{cases}$$

$$\begin{pmatrix} -\frac{19}{3} \\ 3 \end{pmatrix} = 3X_a \begin{pmatrix} 1 \\ -\frac{3}{4} \end{pmatrix} + X_b \begin{pmatrix} 1 \\ -\frac{1}{2} \end{pmatrix} + 2 \begin{pmatrix} -3 \\ 2 \end{pmatrix} + \begin{pmatrix} -\frac{7}{3} \\ 1 \end{pmatrix}$$

$$3X_a + X_b = 2$$

$$-\frac{9}{4} X_a - \frac{1}{2} X_b = -2 \quad \Rightarrow \quad X_a = \frac{-2}{-3/2} = \frac{4}{3}$$

$$X_b = 2 - 3X_a = -2$$

Number of integration constants.

Usually, but not always, equals the sum of the order of the simultaneous equations

…an example that breaks this rule …

$$\frac{\mathrm{d}x}{\mathrm{d}t}+\frac{\mathrm{d}y}{\mathrm{d}t}+y=t$$

$$\frac{\mathrm{d}^2x}{\mathrm{d}t^2}+\frac{\mathrm{d}^2y}{\mathrm{d}t^2}+3x+7y=\mathrm{e}^{2t}.$$

3 integration constants?

$$\frac{d}{dt}\left(\frac{\mathrm{d}x}{\mathrm{d}t}+\frac{\mathrm{d}y}{\mathrm{d}t}+y=t\right)-\left(\frac{\mathrm{d}^2x}{\mathrm{d}t^2}+\frac{\mathrm{d}^2y}{\mathrm{d}t^2}+3x+7y=\mathrm{e}^{2t}\right) \Rightarrow \frac{dy}{dt}-3x-7y=1-e^{2t}$$

$$\frac{\mathrm{d}x}{\mathrm{d}t}+\frac{\mathrm{d}y}{\mathrm{d}t}+y=t$$

$$-\frac{dy}{dt}+3x+7y=-1+e^{2t}$$

$$\Rightarrow \begin{pmatrix} \frac{d}{dt} & \frac{d}{dt}+1 \\ 3 & -\frac{d}{dt}+7 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}=\begin{pmatrix} t \\ e^{2t}-1 \end{pmatrix}$$

Actually only 2 integration constants needed!

An alternative method of solution

$$\frac{\mathrm{d}x}{\mathrm{d}t}+\frac{\mathrm{d}y}{\mathrm{d}t}+y=t \qquad \Rightarrow \frac{\mathrm{d}x}{\mathrm{d}t}=t-\frac{\mathrm{d}y}{\mathrm{d}t}-y$$

$$\frac{dy}{dt}-3x-7y=1-e^{2t}$$

$$\frac{d}{dt}\left(\frac{dy}{dt}-3x-7y=1-e^{2t}\right) \Rightarrow \frac{d^2y}{dt^2}-3\left(\frac{dx}{dt}\right)-7\frac{dy}{dt}=-2e^{2t}$$

An alternative method of solution

$$\frac{dx}{dt}+\frac{dy}{dt}+y=t \qquad \Rightarrow \frac{dx}{dt}=t-\frac{dy}{dt}-y$$

$$\frac{dy}{dt}-3x-7y=1-e^{2t}$$

$$\frac{d}{dt}\left(\frac{dy}{dt}-3x-7y=1-e^{2t}\right) \Rightarrow \frac{d^2y}{dt^2}-3(t-\frac{dy}{dt}-y)-7\frac{dy}{dt}=-2e^{2t}$$

(Only 2 integration constants needed)