

Normal modes

(Matrix method)

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} A_1 \cos(\omega_1 t + \phi_1) + \begin{pmatrix} 1 \\ -1 \end{pmatrix} A_2 \cos(\omega_2 t + \phi_2)$$

$$m\ddot{x} = -mg \frac{x}{l} + k(y - x)$$

$$x + y = 2A_1 \cos(\omega_1 t + \phi_1)$$

$$m\ddot{y} = -mg \frac{y}{l} - k(y - x)$$

$$x - y = 2A_2 \cos(\omega_2 t + \phi_2)$$

Solution II - Decoupling method : (simpler in this case)

$$m(\ddot{x} + \ddot{y}) = -m \frac{g}{l} (x + y)$$

$$\ddot{q}_1 = -\omega_1^2 q_1, \quad q_1 = (x + y), \quad \omega_1^2 = \frac{g}{l}$$

$$q_1 = A_1 \cos(\omega_1 t + \phi_1)$$

centre of mass motion

$$m(\ddot{x} - \ddot{y}) = \left(-m \frac{g}{l} - 2k\right)(x - y)$$

$$\ddot{q}_2 = -\omega_2^2 q_2, \quad q_2 = (x - y), \quad \omega_2^2 = \frac{g}{l} + 2 \frac{k}{m}$$

$$q_2 = A_2 \cos(\omega_2 t + \phi_2)$$

relative motion

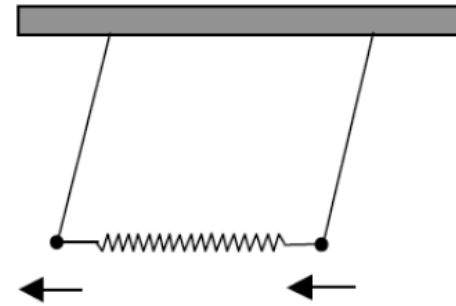
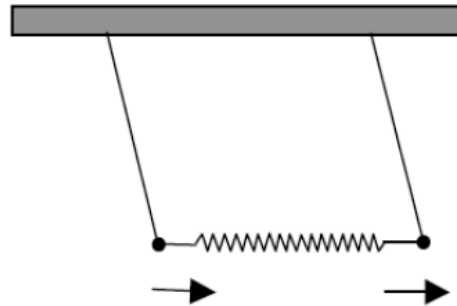
normal coordinates

$$m(\ddot{x} + \ddot{y}) = -m \frac{g}{l}(x + y)$$

$$\ddot{q}_1 = -\omega_1^2 q_1, \quad q_1 = (x + y), \quad \omega_1^2 = \frac{g}{l}$$

$$q_1 = A_1 \cos(\omega_1 t + \phi_1)$$

centre of mass motion

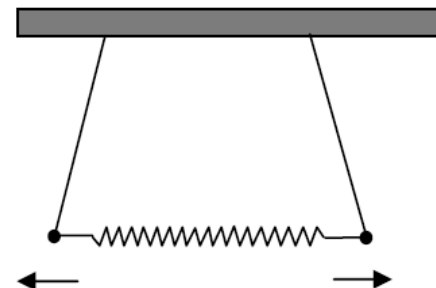
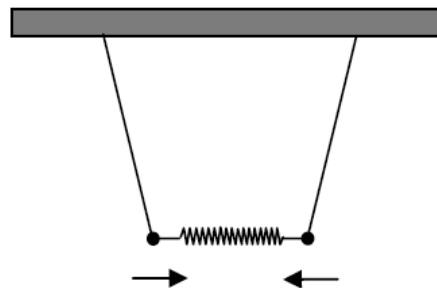


$$m(\ddot{x} - \ddot{y}) = (-m \frac{g}{l} - 2k)(x - y)$$

$$\ddot{q}_2 = -\omega_2^2 q_2, \quad q_2 = (x - y), \quad \omega_2^2 = \frac{g}{l} + 2 \frac{k}{m}$$

$$q_2 = A_2 \cos(\omega_2 t + \phi_2)$$

relative motion

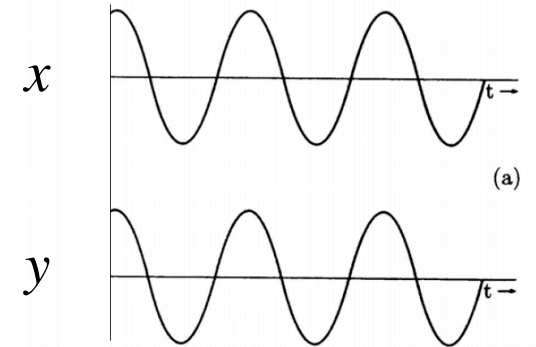


General solution

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} A_1 \cos(\omega_1 t + \phi_1) + \begin{pmatrix} 1 \\ -1 \end{pmatrix} A_2 \cos(\omega_2 t + \phi_2)$$

Constants determined by initial conditions

Ex 1 $x(0) = y(0) = a \quad \dot{x}(0) = \dot{y}(0) = 0$



$$A_1 \cos(\phi_1) + A_2 \cos(\phi_2) = a$$

$$A_1 \cos(\phi_1) - A_2 \cos(\phi_2) = a$$

$$A_1 \omega_1 \sin(\phi_1) + A_2 \omega_2 \sin(\phi_2) = 0$$

$$A_1 \omega_1 \sin(\phi_1) - A_2 \omega_2 \sin(\phi_2) = 0$$

$$\Rightarrow A_2 = 0, A_1 = a, \phi_1 = 0$$

$$\begin{pmatrix} x \\ y \end{pmatrix} = a \cos(\omega_1 t) \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad t > 0$$

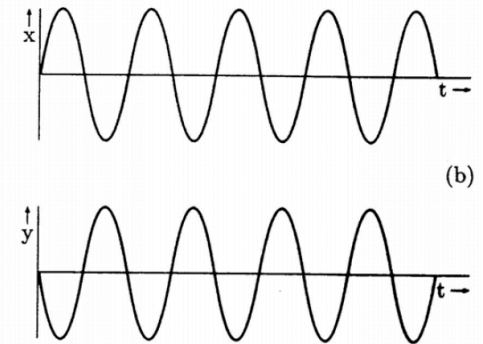
1st Normal mode excitation

General solution

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} A_1 \cos(\omega_1 t + \phi_1) + \begin{pmatrix} 1 \\ -1 \end{pmatrix} A_2 \cos(\omega_2 t + \phi_2)$$

Constants determined by initial conditions

Ex2 $x(0) = y(0) = 0 \quad \dot{x}(0) = -v, \quad \dot{y}(0) = v$



$$A_1 \cos(\phi_1) + A_2 \cos(\phi_2) = 0$$

$$A_1 \cos(\phi_1) - A_2 \cos(\phi_2) = 0$$

$$A_1 \omega_1 \sin(\phi_1) + A_2 \omega_2 \sin(\phi_2) = -v$$

$$A_1 \omega_1 \sin(\phi_1) - A_2 \omega_2 \sin(\phi_2) = v$$

$$\Rightarrow A_1 = 0, \quad \phi_2 = \pi / 2, \quad A_2 = -\frac{v}{\omega_2}$$

$$\begin{pmatrix} x \\ y \end{pmatrix} = \frac{v}{\omega_2} \sin(\omega_1 t) \begin{pmatrix} 1 \\ -1 \end{pmatrix}, \quad t > 0$$

2nd Normal mode excitation

General solution

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} A_1 \cos(\omega_1 t + \phi_1) + \begin{pmatrix} 1 \\ -1 \end{pmatrix} A_2 \cos(\omega_2 t + \phi_2)$$

Constants determined by initial conditions

Ex3 $x(0) = a \quad y(0) = 0 \quad \dot{x}(0) = \dot{y}(0) = 0$

$$A_1 \cos(\phi_1) + A_2 \cos(\phi_2) = a$$

$$A_1 \cos(\phi_1) - A_2 \cos(\phi_2) = 0$$

$$A_1 \omega_1 \sin(\phi_1) + A_2 \omega_2 \sin(\phi_2) = 0$$

$$A_1 \omega_1 \sin(\phi_1) - A_2 \omega_2 \sin(\phi_2) = 0$$

$$\Rightarrow \phi_1 = \phi_2 = 0, \quad A_1 = A_2 = \frac{a}{2}$$

$$\begin{pmatrix} x \\ y \end{pmatrix} = \frac{a}{2} \cos(\omega_1 t) \begin{pmatrix} 1 \\ 1 \end{pmatrix} + \frac{a}{2} \cos(\omega_2 t) \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

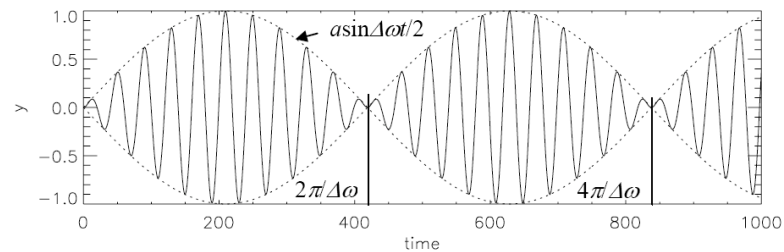
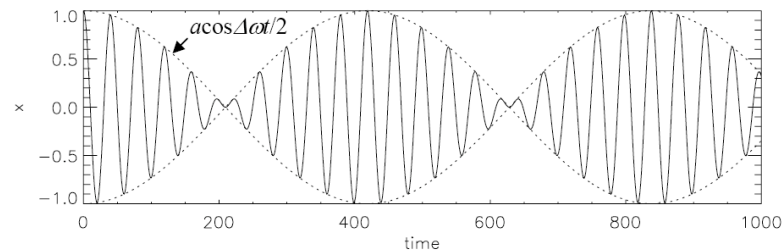
both modes excited

$$\begin{pmatrix} x \\ y \end{pmatrix} = \frac{a}{2} \cos(\omega_1 t) \begin{pmatrix} 1 \\ 1 \end{pmatrix} + \frac{a}{2} \cos(\omega_2 t) \begin{pmatrix} 1 \\ -1 \end{pmatrix} \quad \text{both modes excited}$$

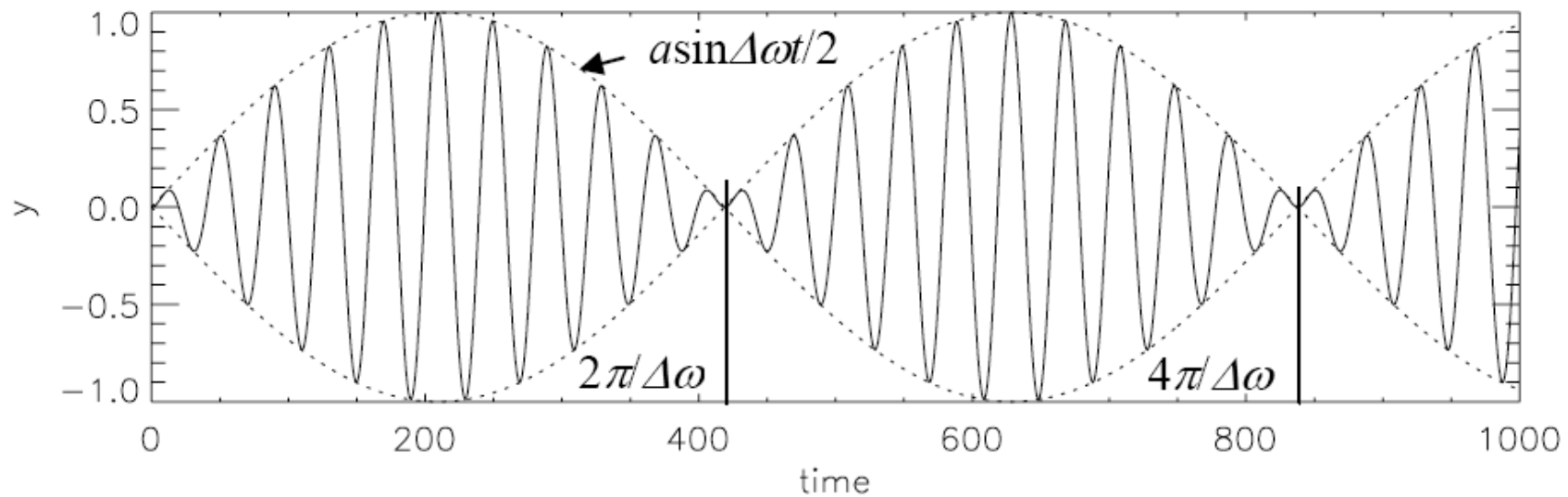
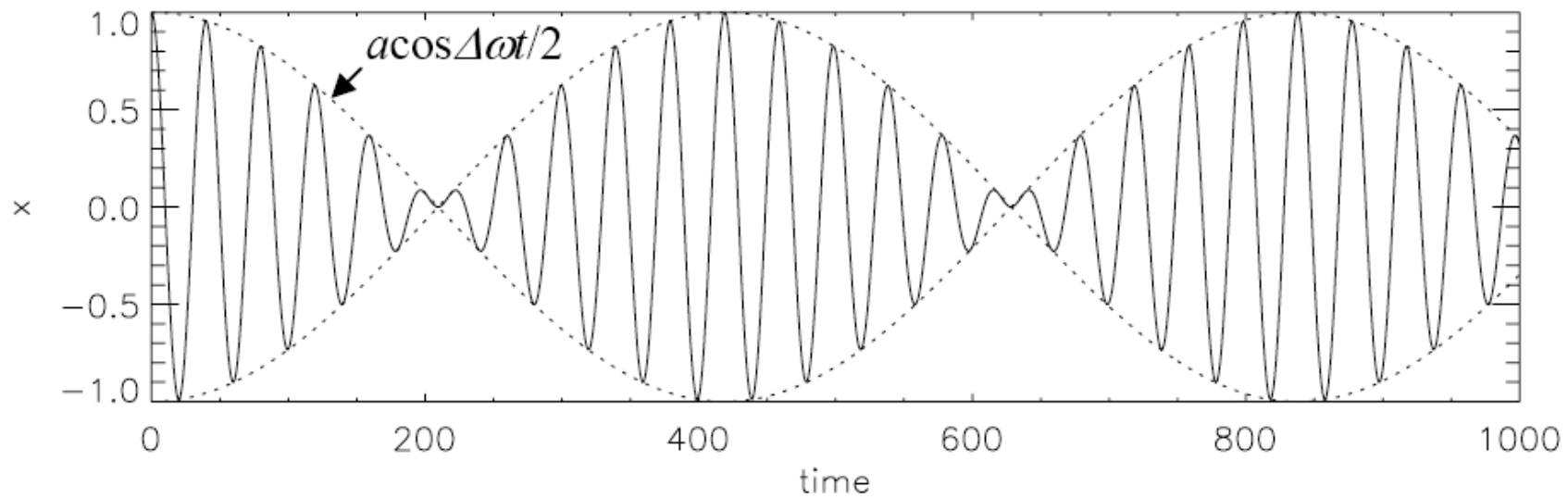
$$x = a \cos\left(\frac{\omega_1 + \omega_2}{2} t\right) \cos\left(\frac{\omega_1 - \omega_2}{2} t\right) = a \cos(\bar{\omega} t) \cos(\Delta\omega t / 2)$$

$$y = a \sin\left(\frac{\omega_1 + \omega_2}{2} t\right) \sin\left(\frac{\omega_1 - \omega_2}{2} t\right) = a \sin(\bar{\omega} t) \sin(\Delta\omega t / 2)$$

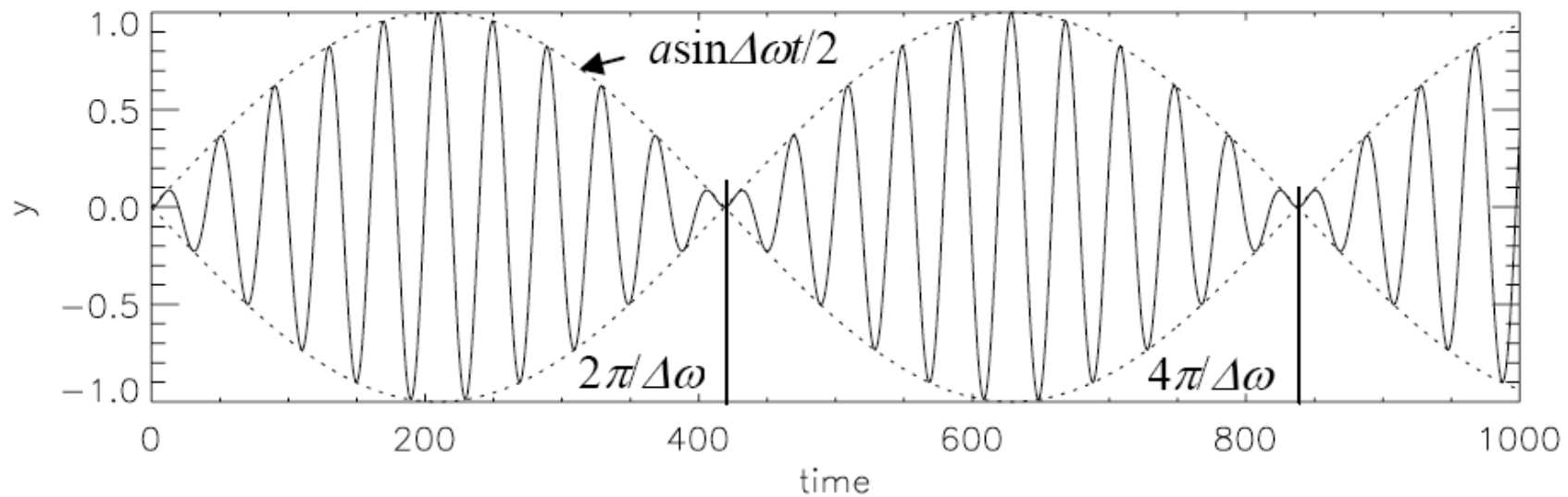
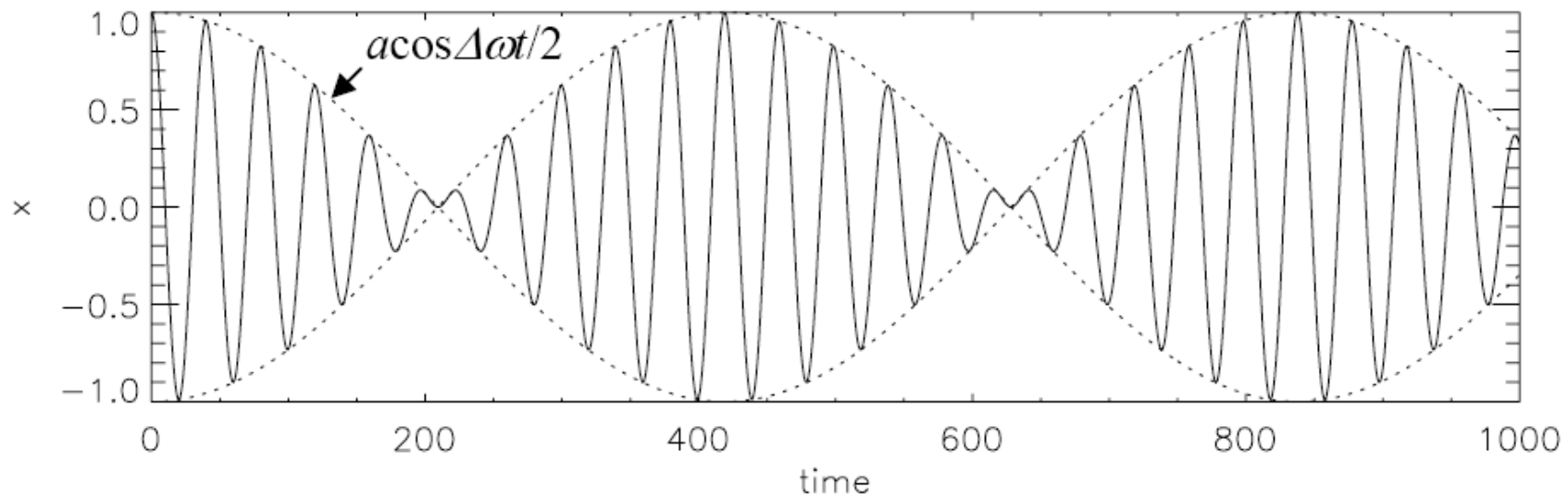
where $\bar{\omega} = \frac{\omega_1 + \omega_2}{2}$ and $\Delta\omega = \omega_1 - \omega_2$



$$\Delta\omega \ll \bar{\omega}$$



$$\begin{aligned}
 x &= a \cos\left(\frac{\omega_1 + \omega_2}{2} t\right) \cos\left(\frac{\omega_1 - \omega_2}{2} t\right) &= a \cos(\bar{\omega} t) \cos(\Delta\omega t / 2) \\
 y &= a \sin\left(\frac{\omega_1 + \omega_2}{2} t\right) \sin\left(\frac{\omega_1 - \omega_2}{2} t\right) &= a \sin(\bar{\omega} t) \sin(\Delta\omega t / 2)
 \end{aligned}$$



Note transfer of energy between the two pendula

“Beats”

Energy of motion

P.E. V + K.E. K (Potential energy+Kinetic energy)

$$m\ddot{x} = -mg \frac{x}{l} + k(y - x) = -\frac{\partial V}{\partial x}$$

Conservative force

Energy of motion

P.E. V + K.E. K

$$m\ddot{x} = -mg\frac{x}{l} + k(y-x) = -\frac{\partial V}{\partial x} \quad V(x,y) = mg\frac{x^2}{2l} + \frac{1}{2}kx^2 - kxy + f(y)$$

$$m\ddot{y} = -mg\frac{y}{l} - k(y-x) = -\frac{\partial V}{\partial y} = kx - \frac{df}{dy} \quad \Rightarrow f(y) = mg\frac{y^2}{2l} + \frac{1}{2}ky^2 + C$$

$$V(x,y) = \frac{1}{2}m\left(\frac{g}{l} + \frac{k}{m}\right)(x^2 + y^2) - kxy + C$$

$$K = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2)$$

In normal coordinates:

$$q_1 = \frac{1}{\sqrt{2}}(x+y), \quad q_2 = \frac{1}{\sqrt{2}}(x-y) \quad \dots \text{with } \frac{1}{\sqrt{2}} q_1, q_2 \text{ rotated axis w.r.t. } x, y$$

$$x = \frac{1}{\sqrt{2}}(q_1 + q_2), \quad y = \frac{1}{\sqrt{2}}(q_1 - q_2)$$

Energy of motion

P.E. V + K.E. K

$$m\ddot{x} = -mg\frac{x}{l} + k(y-x) = -\frac{\partial V}{\partial x} \quad V(x,y) = mg\frac{x^2}{2l} + \frac{1}{2}kx^2 - kxy + f(y)$$

$$m\ddot{y} = -mg\frac{y}{l} - k(y-x) = -\frac{\partial V}{\partial y} = kx - \frac{df}{dy} \quad \Rightarrow f(y) = mg\frac{y^2}{2l} + \frac{1}{2}ky^2 + C$$

$$V(x,y) = \frac{1}{2}m\left(\frac{g}{l} + \frac{k}{m}\right)(x^2 + y^2) - kxy + C$$

$$K = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2)$$

In normal coordinates:

$$q_1 = \frac{1}{\sqrt{2}}(x+y), \quad q_2 = \frac{1}{\sqrt{2}}(x-y)$$

$$x = \frac{1}{\sqrt{2}}(q_1 + q_2), \quad y = \frac{1}{\sqrt{2}}(q_1 - q_2)$$

$$\begin{aligned} V &= \frac{1}{2}m\frac{g}{l}q_1^2 + \frac{1}{2}m\left(\frac{g}{l} + \frac{2k}{m}\right)q_2^2 \\ &= \frac{1}{2}m\omega_1^2q_1^2 + \frac{1}{2}m\omega_2^2q_2^2 \end{aligned}$$

$$K = \frac{1}{2}m(\dot{q}_1^2 + \dot{q}_2^2)$$

$$V = \frac{1}{2}m\frac{g}{l}q_1^2 + \frac{1}{2}m\left(\frac{g}{l} + \frac{2k}{m}\right)q_2^2$$

$$= \frac{1}{2}m\omega_1^2q_1^2 + \frac{1}{2}m\omega_2^2q_2^2$$

$$\ddot{q}_1 = -\omega_1^2q_1, \quad q_1 = \frac{1}{\sqrt{2}}(x+y), \quad \omega_1^2 = \frac{g}{l}$$

$$K = \frac{1}{2}m(\dot{q}_1^2 + \dot{q}_2^2)$$

$$\ddot{q}_2 = -\omega_2^2q_2, \quad q_2 = \frac{1}{\sqrt{2}}(x-y), \quad \omega_2^2 = \frac{g}{l} + 2\frac{k}{m}$$

$$E = \left(\frac{1}{2}m\omega_1^2q_1^2 + \frac{1}{2}m\dot{q}_1^2\right) + \left(\frac{1}{2}m\omega_2^2q_2^2 + \frac{1}{2}m\dot{q}_2^2\right)$$

$$= E_1 + E_2$$

Total energy = sum of energies of the normal modes

Parseval's theorem