Fundamental principles of particle physics

Our description of the fundamental interactions and particles rests on two fundamental structures :

- Quantum Mechanics
- Symmetries

Symmetries

Central to our description of the fundamental forces :

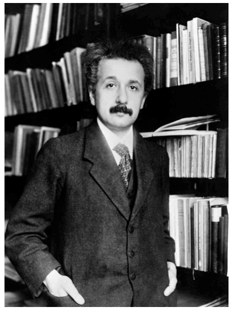
Relativity - translations and Lorentz transformations

Lie symmetries - $SU(3) \otimes SU(2) \otimes U(1)$

Copernican principle : “Your system of co-ordinates and units is nothing special”

⇒ Physics independent of system choice

Special relativity



- Space time point $a^{\mu} = (ct, x, y, z)$ not invariant under translations

- Space-time vector $(a+\Delta a)^{\mu} - a^{\mu} = \Delta a^{\mu} = (c\Delta t, \Delta x, \Delta y, \Delta z)$

  Invariant under translations …but not invariant under rotations or boosts

- Einstein postulate : the real invariant distance is

$$\left(\Delta a^0\right)^2 - \left(\Delta a^1\right)^2 - \left(\Delta a^2\right)^2 - \left(\Delta a^3\right)^2 = \sum_{\mu,\nu=0}^{3} g_{\mu\nu} \Delta a^{\mu} \Delta a^{\nu} = \Delta a^{\mu} \Delta a_{\mu} = \left(\Delta a\right)^2$$

$$g_{\mu\nu} = diag(+1, -1, -1, -1)$$

- Physics invariant under all transformations that leave all such distances invariant :

  Translations and Lorentz transformations

Fundamental principles of particle physics

Quantum Mechanics

+

Relativity

} Quantum Field theory

Relativistic quantum field theory

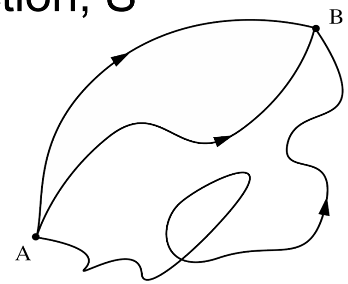
Fundamental division of physicist's world :

speed

Action (S)	slow	fast
large	Classical Newton	Classical relativity
small	Classical Quantum mechanics	Quantum Field theory

→ c

Action, S



$$S = \int_{t_A}^{t_B} (K.E. - P.E.)\,dt$$

Action $S = \int_{t_1}^{t_2} L\, dt$

Lagrangian $L = T - V$ (Nonrelativistic mechanics)

- Quantum mechanics … sum over all paths with amplitude $\propto e^{iS/\hbar}$

- Classical path … minimises action

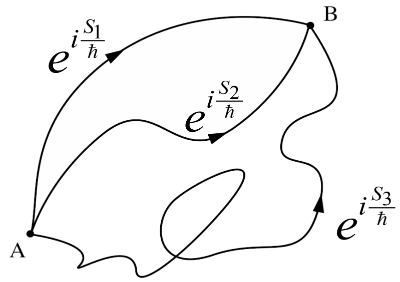
"Principle of Least Action"
Feynman Lectures in Physics
Vol II Chapter 19

(Lagrangian invariant under all the symmetries of nature

-makes it easy to construct viable theories)

Relativistic quantum field theory

Fundamental division of physicist's world :



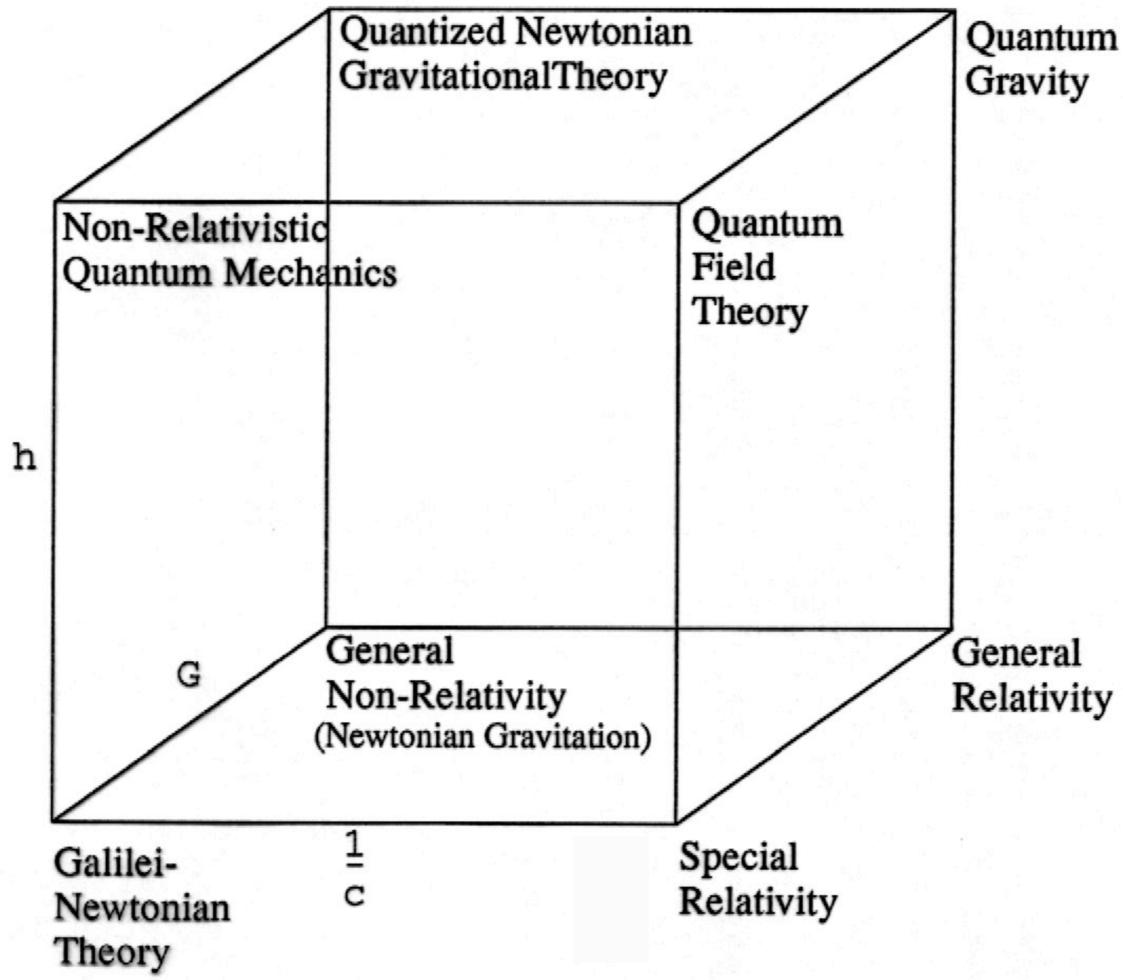
speed

Action (S)		slow	fast
	large	Classical Newton	Classical relativity
	small	Classical Quantum mechanics	Quantum Field theory

$\hbar$ $(QM \text{ amplitude} \propto e^{i(\frac{S}{\hbar})})$

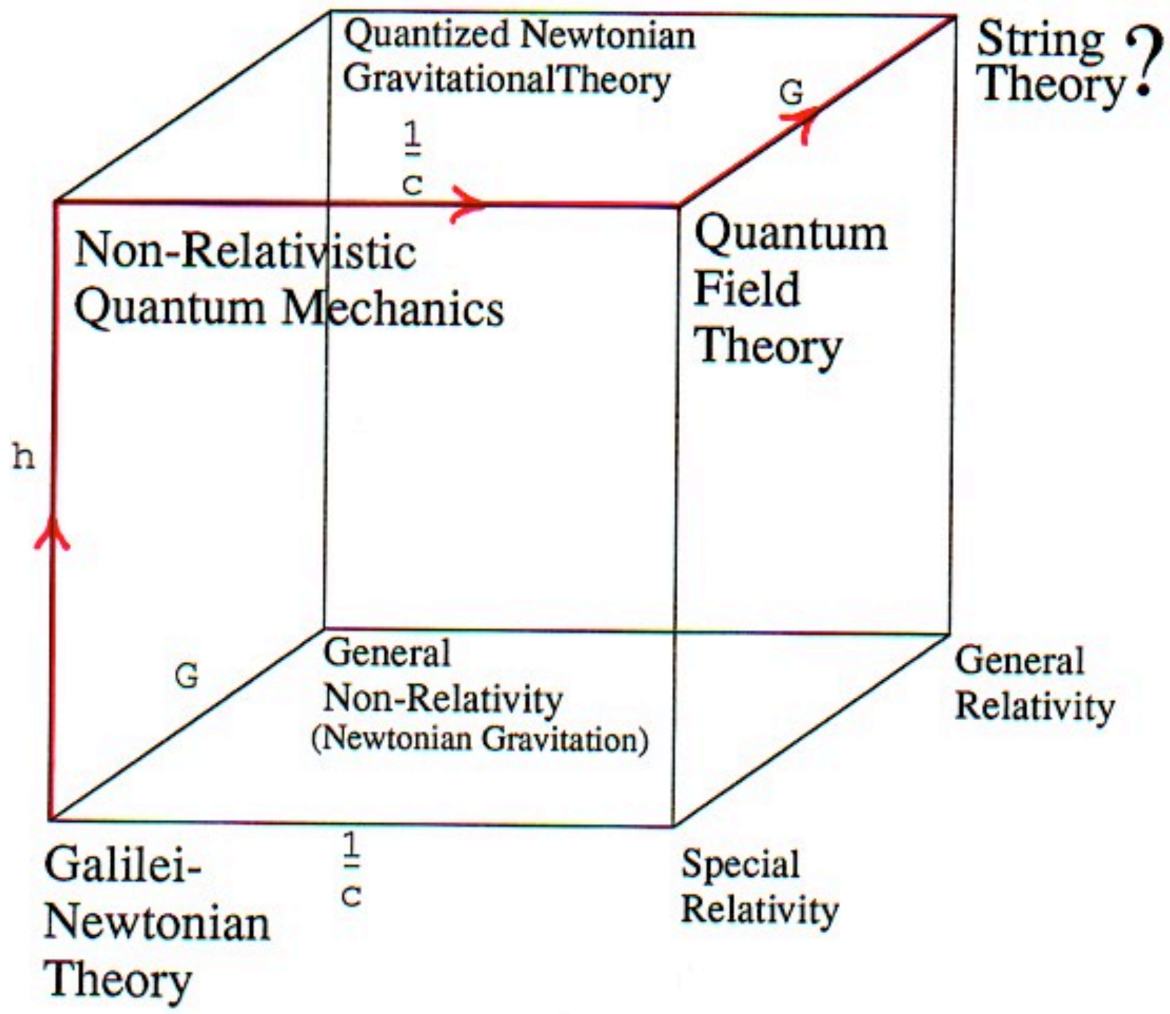
c

Bronshtein's 'cube of theories'



Matvei Petrovich Bronshtein (1906-38)
Progress in Astronomical Sciences (Gostekhizdat, Moscow, 1933), Vol. 3, p. 3
… also Gamov, Ivanenko & Landau, Zh. Russ. Fiz.-Khim. O-va., Chast Fiz. 60, 13 (1928)

Bronshtein's 'cube of theories'



Why Quantum field theory?

Quantum Mechanics : Quantization of dynamical system of particles

Quantum Field Theory : Application of QM to dynamical system of fields

- No right to assume that any relativistic process can be explained by single particle since $E=mc^2$ allows pair creation

- (Relativistic) QM has physical problems. For example it violates causality

$$U(t) = \langle x | e^{-i(p^2/2m)t} | x_0 \rangle = \int d^3p \int d^3p' \langle x | p \rangle \langle p | e^{-i(p^2/2m)t} | p' \rangle \langle p' | x_0 \rangle$$

$$= \frac{1}{2\pi^3} \int d^3p \, e^{-i(p^2/2m)t} \, e^{ip(x-x_0)}$$

$$= \left(\frac{m}{2\pi i t} \right)^{3/2} e^{im(x-x_0)^2/2t} \quad \text{... nonzero for all } x,\ t$$

Amplitude for free propagation from x_0 to x

Relativistic case : $U(t) \propto e^{-m\sqrt{x^2 - t^2}}$ ... nonzero for all $x,\ t$

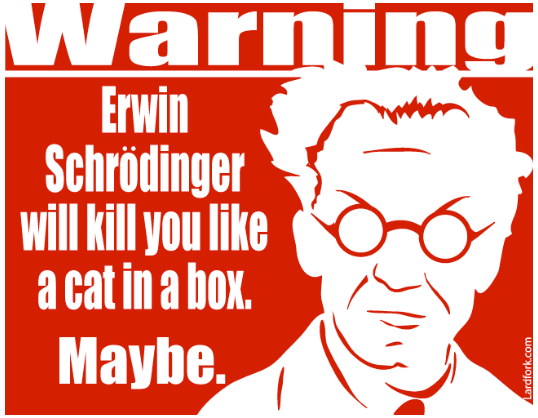
Quantum Mechanics

$$E - \frac{p^2}{2m} = 0$$

Classical – non relativistic

$$i\hbar \frac{\partial \phi}{\partial t} + \frac{\hbar^2}{2m} \nabla^2 \phi = 0$$

Quantum Mechanical : Schrodinger eq



Quantum Mechanics

$E - \frac{p^2}{2m} = 0$	Classical – non relativistic
$i\hbar \frac{\partial \phi}{\partial t} + \frac{\hbar^2}{2m} \nabla^2 \phi = 0$	Quantum Mechanical : Schrodinger eq

$$E^2 - \mathbf{p}^2 = m^2$$

Classical – relativistic (Natural units $\hbar = c = 1$)

$$-\frac{\partial^2 \phi}{\partial t^2} + \nabla^2 \phi = m^2 \phi$$

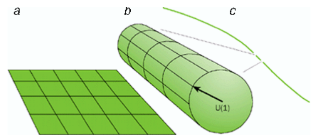
Quantum Mechanical - relativistic :

Klein-Gordon (Schrodinger) equation

(natural units)

• Relativistic QM - The Klein Gordon equation (1926)

Scalar particle (field) (J=0) : $\phi(\mathbf{x})$



$$E^2 = \mathbf{p}^2 + m^2 \quad \Rightarrow \quad -\frac{\partial^2 \phi}{\partial t^2} + \nabla^2 \phi = m^2 \phi$$

(natural units)

Energy eigenvalues $E = \pm(\mathbf{p}^2 + m^2)^{1/2}$???

1927 Dirac tried to eliminate negative solutions by writing a relativistic equation linear in E (a theory of fermions)

1934 Pauli and Weisskopf revived KG equation with E<0 solutions as E>0 solutions for particles of opposite charge (antiparticles). Unlike Dirac's hole theory this interpretation is applicable to bosons (integer spin) as well as to fermions (half integer spin).

As we shall see the antiparticle states make the field theory causal

Physical interpretation of Quantum Mechanics

Schrödinger equation (S.E.)

$$i\frac{\partial\phi}{\partial t}+\frac{1}{2m}\nabla^2\phi=0$$

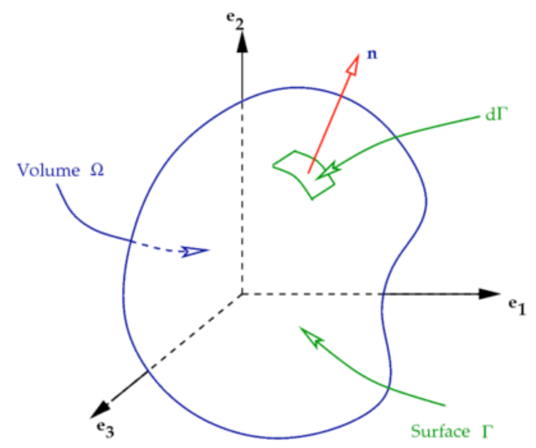
$$i\phi^*(S.E.)-i\phi(S.E.)^* \Rightarrow \frac{\partial\rho}{\partial t}+\nabla.\mathbf{j}=0 \text{ continuity eq.}$$

$$\rho=|\phi|^2$$

"probability density"

$$\mathbf{j}=-\frac{i}{2m}(\phi^*\nabla\phi-\phi\nabla\phi^*)$$

"probability current"



Divergence Theorem

$$\iiint_\Omega \nabla.\mathrm{M}\ \mathrm{dV}=\iint_\Gamma M.n\ dS$$

Physical interpretation of Quantum Mechanics

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"probability current"

Klein Gordon equation

$$-\frac{\partial^2\phi}{\partial t^2}+\nabla^2\phi=m^2\phi$$

$$\rho=2E|N|^2$$

$$\rho=i\left(\phi^*\frac{\partial\phi}{\partial t}-\phi\frac{\partial\phi^*}{\partial t}\right)$$

$$\mathbf{j}=-i(\phi^*\nabla\phi-\phi\nabla\phi^*)$$

$$j^\mu=(\rho,\mathbf{j})$$

Negative probability?

$$\phi=Ne^{-ip.x},\quad \rho=2E|N|^2$$

$$\int_V\rho\,dV=\int\rho\,d^3x=2E$$

$$f_p^{\pm}=e^{\mp ip.x}\frac{1}{\sqrt{2p^0V}}$$

Normalised free particle solutions

Lorentz transformations :

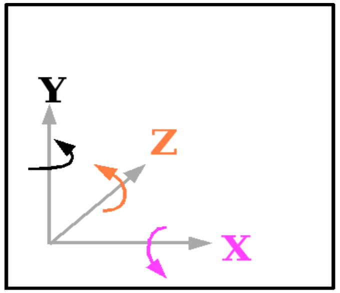
$g_{\mu\nu} = Diagonal(1,-1,-1,-1)$

$$x^\mu \to \Lambda^\mu_\nu x^\nu = \sum_{\nu=0}^{3} \Lambda^\mu_\nu x^\nu = x'^\mu \quad \Rightarrow \quad g_{\mu\nu} x'^\mu x'^\nu = g_{\mu\nu} x^\mu x^\nu \quad \Rightarrow \quad g_{\mu\nu} \Lambda^\mu_\alpha \Lambda^\nu_\beta = g_{\alpha\beta}$$

(Summation assumed)

Solutions :

- 3 rotations R



$$R_z(\theta)\begin{pmatrix} ct \\ x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta & \sin\theta & 0 \\ 0 & -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} ct \\ x \\ y \\ z \end{pmatrix} = \begin{pmatrix} ct \\ x\cos\theta + y\sin\theta \\ y\cos\theta - x\sin\theta \\ z \end{pmatrix}$$

Lorentz transformations :

$$x^\mu \rightarrow \Lambda^\mu_\nu x^\nu = \sum_{\nu=0}^{3} \Lambda^\mu_\nu x^\nu = x'^\mu \quad \Rightarrow \quad g_{\mu\nu} x'^\mu x'^\nu = g_{\mu\nu} x^\mu x^\nu \quad \Rightarrow \quad g_{\mu\nu} \Lambda^\mu_\alpha \Lambda^\nu_\beta = g_{\alpha\beta}$$

(Summation assumed)

Solutions :

- 3 rotations R

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta & \sin\theta & 0 \\ 0 & -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

- 3 boosts B

$$\begin{pmatrix} \cosh\alpha & \sinh\alpha & 0 & 0 \\ \sinh\alpha & \cosh\alpha & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

- Space reflection – parity P

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

- Time reflection, time reversal T

$$\begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

- **The Lorentz transformations form the group, G†, SO(3,1)** $(g_1 g_2 \in G \quad if \quad g_1, g_2 \in G)$

Rotations Can represent group element in terms of "generators" of an algebra

$$R(\theta) = e^{-i\mathbf{J}.\theta/\hbar}, \quad J_z = i\hbar(x\tfrac{\partial}{\partial y} - y\tfrac{\partial}{\partial x})..$$

Angular momentum operator

$$(c.f.\, \mathbf{J} = \mathbf{r} \times \mathbf{p})$$

$$[J_i, J_j] = i\hbar \sum_{k=1}^{3} \varepsilon_{ijk} J_k$$

Lie algebra SU(2)

ε_{ijk} totally antisymmetric Levi-Civita symbol,

$$\varepsilon_{123} = \varepsilon_{231} = \varepsilon_{312} = +1; \quad \varepsilon_{213} = \varepsilon_{132} = \varepsilon_{321} = -1$$

The J_i are the "generators" of the group. SO(3) (SU(2))

Their commutation relations define a "Lie algebra"†.

Derivation of the commutation relations of SO(3) (SU(2))

$R_x(\varepsilon)R_y(\eta)R_x(-\varepsilon)R_y(-\eta)$ ε, η *small* (infinitesimal)

$$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & \varepsilon \\ 0 & -\varepsilon & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & \eta \\ 0 & 1 & 0 \\ -\eta & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -\varepsilon \\ 0 & \varepsilon & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & \eta \\ 0 & 1 & 0 \\ -\eta & 0 & 1 \end{pmatrix}$$

$$\approx \begin{pmatrix} 1 & \varepsilon\eta & 0 \\ -\varepsilon\eta & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = R_z(\varepsilon\eta) \approx (1 - i\varepsilon\eta J_z)$$

$$R_x(\varepsilon)R_y(\eta)R_x(-\varepsilon)R_y(-\eta)$$

$$= (1 - i\varepsilon J_x)(1 - i\varepsilon J_y)(1 + i\varepsilon J_x)(1 + i\varepsilon J_y) = -\varepsilon\eta(J_x J_y - J_y J_x)$$

Equating the two equations implies

$$[J_x, J_y] = iJ_z$$

QED

Demonstration that $R_Z(\theta) = e^{-iJ_z\theta}$

$J_z = i\hbar(x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x})$..

$$R_Z(\theta)\psi(x,y) \equiv R_Z(\theta)\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x' \\ y' \end{pmatrix} \equiv \psi(x', y')$$

For small θ, $\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} x + \theta y \\ y - \theta x \end{pmatrix}$

$$R_Z(\theta)\psi(x,y) = \psi(x+\theta y, y - \theta x) \approx \psi(x,y) + \theta(y\frac{\partial \psi}{\partial x} - x\frac{\partial \psi}{\partial y})$$
$$= (1 - i\theta(xp_y - yp_x))\psi(x,y)$$

$$i.e.\ R_Z(\theta) \approx (1 - i\theta(xp_y - yp_x)) = 1 - i\theta J_z$$

For large θ

$$R_Z(\theta = n\varepsilon) = e^{-iJ_z\theta}$$

- **The Lorentz transformations form the group, G , SO(3,1)**

Rotations The matrix “representation” of J acting on a four vector

$$R(\theta) = e^{-i\mathbf{J}.\theta/\hbar}$$

$$\begin{pmatrix} ct' \\ x' \\ y' \\ z' \end{pmatrix} = R_z(\theta)\begin{pmatrix} ct \\ x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta & \sin\theta & 0 \\ 0 & -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} ct \\ x \\ y \\ z \end{pmatrix}$$

$$(c.f.\ \mathbf{J} = \mathbf{r}\times\mathbf{p})$$

$$R_z(\theta) = e^{-iJ_z\theta/\hbar} = I - iJ_z\theta/\hbar + \frac{1}{2}(-iJ_z\theta/\hbar).(-iJ_z\theta/\hbar) + \ldots$$

$$-iJ_z/\hbar = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \qquad (-iJ_z/\hbar)^2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$R_z(\theta) = e^{-iJ_z\theta/\hbar} = \begin{pmatrix} 1 & & & \\ & 1+\frac{\theta^2}{2}+\ldots & \theta+\ldots & \\ & -\theta+\ldots & 1+\frac{\theta^2}{2}+\ldots & \\ & & & 1 \end{pmatrix}$$