Exponential h(x)

$$Lf = a_2 \frac{\mathrm{d}^2 f}{\mathrm{d}x^2} + a_1 \frac{\mathrm{d}f}{\mathrm{d}x} + a_0 f = h(x) = H\mathrm{e}^{\gamma x}$$

$$\equiv a_2 \left(\frac{\mathrm{d}}{\mathrm{d}x} - \alpha_1\right)\left(\frac{\mathrm{d}}{\mathrm{d}x} - \alpha_2\right) f = H\mathrm{e}^{\gamma x}.$$

$$\mathrm{CF} = A_{1,2}\mathrm{e}^{\alpha_{1,2} x}.$$

PI : $$f = \frac{H\mathrm{e}^{\gamma x}}{a_2(\gamma - \alpha_1)(\gamma - \alpha_2)}$$

$$\alpha_1 \neq \alpha_2 \neq \ \gamma \neq \alpha_1$$

$$f = \frac{Hx\mathrm{e}^{\alpha_2 x}}{a_2(\alpha_2 - \alpha_1)}$$

$$\gamma = \alpha_2 \neq \alpha_1$$

$$\mathrm{CF} = A\mathrm{e}^{\alpha x} + Bxe^{\alpha x}$$

$$\alpha_1 = \alpha_2 = \alpha = \gamma$$

PI : *Try* $f = Px^2\mathrm{e}^{\alpha x}$

$$a_2\left(\frac{\mathrm{d}}{\mathrm{d}x}-\alpha_1\right)\left(\frac{\mathrm{d}}{\mathrm{d}x}-\alpha_2\right)f = H\mathrm{e}^{\gamma x}.$$

$$Try \quad f = Px^2\mathrm{e}^{\alpha x}$$

$$a_2\left(\frac{\mathrm{d}}{\mathrm{d}x}-\alpha\right)^2 Px^2\mathrm{e}^{\alpha x} = a_2\left(\frac{\mathrm{d}}{\mathrm{d}x}-\alpha\right)2Px\mathrm{e}^{\alpha x} = 2a_2P\mathrm{e}^{\alpha x} \quad = H\mathrm{e}^{\alpha x}$$

$$\Rightarrow \boxed{P = \frac{H}{2a_2}}$$

Exponential h(x)

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$$f = \frac{Hx\mathrm{e}^{\alpha_2 x}}{a_2(\alpha_2 - \alpha_1)}$$

$$\gamma = \alpha_2 \neq \alpha_1$$

$$f = \frac{H}{2a_2} x^2 \mathrm{e}^{\alpha x}$$

$$\alpha_1 = \alpha_2 = \alpha = \gamma$$

Ex 4

$$f'' + 3f' + 2f = \mathrm{e}^{-x}, \quad f(0) = 1, \quad f'(0) = 1$$

The CF is $f_0 = A\mathrm{e}^{-2x} + B\mathrm{e}^{-x}$

PI ... try $f_1 = Px\mathrm{e}^{-x}$

$$\mathrm{e}^{-x} = (\frac{\mathrm{d}}{\mathrm{d}x} + 2)(\frac{\mathrm{d}}{\mathrm{d}x} + 1)Px\mathrm{e}^{-x} = (\frac{\mathrm{d}}{\mathrm{d}x} + 2)P\mathrm{e}^{-x}$$

$$= P\mathrm{e}^{-x}.$$

$$P = 1 \quad \text{and} \quad f_1 = x\mathrm{e}^{-x}$$

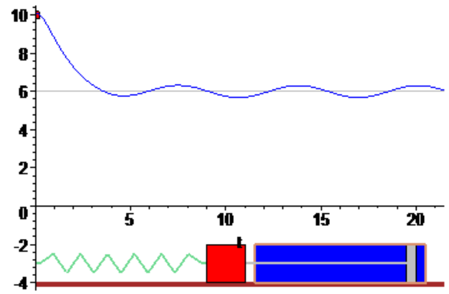
Full solution : $f = f_0 + f_1 = x\mathrm{e}^{-x} + A\mathrm{e}^{-2x} + B\mathrm{e}^{-x}$

Initial conditions $\Rightarrow$ $f = x\mathrm{e}^{-x} - \mathrm{e}^{-2x} + 2\mathrm{e}^{-x}$

Oscillators

$$m\ddot{x} = -m\omega_0^2 x - m\gamma\dot{x} + mF\cos\omega t.$$

spring — friction — forcing



The associated complex equation is

$$\ddot{z} + \gamma\dot{z} + \omega_0^2 z = F\mathrm{e}^{\mathrm{i}\omega t}.$$

Transients :

CF - Auxiliary equation : $z = e^{\alpha t}$.

$\omega_0^2 - \frac{1}{4}\gamma^2 > 0$

$$\alpha^2 + \gamma\alpha + \omega_0^2 = 0 \quad\Rightarrow\quad \alpha = -\tfrac{1}{2}\gamma \pm \mathrm{i}\sqrt{\omega_0^2 - \tfrac{1}{4}\gamma^2}$$

$$= -\tfrac{1}{2}\gamma \pm \mathrm{i}\omega_\gamma \quad \text{where} \quad \omega_\gamma \equiv \omega_0\sqrt{1 - \tfrac{1}{4}\gamma^2/\omega_0^2}.$$

Complementary function

Constant phase “shift”

$$x = \mathrm{e}^{-\gamma t/2}[A\cos(\omega_\gamma t) + B\sin(\omega_\gamma t)] = \mathrm{e}^{-\gamma t/2} N\cos(\omega_\gamma t + \psi)$$

Since $\gamma > 0$, the CF $\rightarrow 0$ as $t \rightarrow \infty$ .... CF describes "transients"

Steady state solutions

$$\ddot{z} + \gamma\dot{z} + \omega_0^2 z = F\mathrm{e}^{i\omega t}.$$

No damping exponential

Particular integral

$$x = \Re\mathrm{e}\left(\frac{F\mathrm{e}^{i\omega t}}{\omega_0^2 - \omega^2 + i\omega\gamma}\right).$$

i.e. the Particular integral describes the steady state solution after the transients have died away.

Since the denominator $= \sqrt{(\omega_0^2 - \omega^2)^2 + \omega^2\gamma^2}\ \mathrm{e}^{i\phi}$ where $\phi \equiv \arctan(\frac{\omega\gamma}{\omega_0^2 - \omega^2})$ the particular integral can be written as

$$x = \frac{F\Re\mathrm{e}(\mathrm{e}^{i(\omega t - \phi)})}{\sqrt{(\omega_0^2 - \omega^2)^2 + \omega^2\gamma^2}} = \frac{F\cos(\omega t - \phi)}{\sqrt{(\omega_0^2 - \omega^2)^2 + \omega^2\gamma^2}}.$$

For $\phi>0$, x achieves the same phase as F at t greater by $\left(\frac{\phi}{\omega}\right)$

- ϕ is called the "phase lag" of the response.

$$x = \frac{F \Re \mathrm{e}(\mathrm{e}^{\mathrm{i}(\omega t - \phi)})}{\sqrt{(\omega_0^2 - \omega^2)^2 + \omega^2 \gamma^2}} = \frac{F \cos(\omega t - \phi)}{\sqrt{(\omega_0^2 - \omega^2)^2 + \omega^2 \gamma^2}}$$

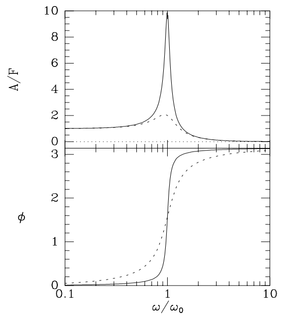
The amplitude of the response is $A = \dfrac{F}{\sqrt{(\omega_0^2 - \omega^2)^2 + \omega^2 \gamma^2}}$,

This has a maximum when $0 = \dfrac{\mathrm{d}A^{-2}}{\mathrm{d}\omega} \propto -4(\omega_0^2 - \omega^2)\omega + 2\omega\gamma^2 \quad \Rightarrow \quad \omega^2 = \omega_0^2 - \tfrac{1}{2}\gamma^2.$

$\omega_R \equiv \sqrt{\omega_0^2 - \gamma^2/2}$ is called the “resonant” frequency

The frictional coefficient causes the resonant frequency to be less than the normal frequency

$$\phi \equiv \arctan\left(\frac{\omega\gamma}{\omega_0^2 - \omega^2}\right)$$



Oscillators $$m\ddot{x} = -\underbrace{m\omega_0^2 x}_{\text{spring}} - \underbrace{m\gamma\dot{x}}_{\text{friction}} + \underbrace{mF\cos\omega t}_{\text{forcing}}.$$

$$\mathrm{F} = mF\cos\omega t, \qquad x = \frac{F\Re(\mathrm{e}^{\mathrm{i}(\omega t-\phi)})}{\sqrt{(\omega_0^2-\omega^2)^2+\omega^2\gamma^2}} = \frac{F\cos(\omega t-\phi)}{\sqrt{(\omega_0^2-\omega^2)^2+\omega^2\gamma^2}}$$

Power Input (steady state) $$P = \frac{\partial W}{\partial t} = \mathrm{F}\dot{x} \qquad W = \int_{x(t_0)}^{x(t)} F\,dx'$$

$$P = \mathrm{F}\dot{x} = mF\cos\omega t \times \frac{-F\omega\sin(\omega t-\phi)}{\sqrt{(\omega_0^2-\omega^2)^2+\omega^2\gamma^2}}$$

$$= \frac{\omega m F^2}{\sqrt{(\omega_0^2-\omega^2)^2+\omega^2\gamma^2}}[-\cos(\omega t)\sin(\omega t-\phi)]$$

$$= -\frac{\frac{1}{2}\omega m F^2}{\sqrt{(\omega_0^2-\omega^2)^2+\omega^2\gamma^2}}[\sin(2\omega t-\phi)+\sin(-\phi)].$$

Average over a period

$$\overline{P} = \frac{\frac{1}{2}\omega m F^2\sin\phi}{\sqrt{(\omega_0^2-\omega^2)^2+\omega^2\gamma^2}}.$$

Energy dissipated

$$m\ddot{x} = -\underset{\text{spring}}{m\omega_0^2 x} - \underset{\text{friction}}{m\gamma\dot{x}} + \underset{\text{forcing}}{mF\cos\omega t}.$$

$$x = \frac{F\cos(\omega t - \phi)}{\sqrt{(\omega_0^2 - \omega^2)^2 + \omega^2\gamma^2}}$$

$$\overline{D} = m\gamma\overline{\dot{x}\dot{x}} = \frac{m\gamma\omega^2 F^2 \tfrac{1}{2}}{(\omega_0^2 - \omega^2)^2 + \omega^2\gamma^2}.$$

$$\overline{\sin^2(\omega t - \phi)} = \frac{1}{2}$$

$$\overline{D} = \overline{P} = \frac{\tfrac{1}{2}\omega m F^2 \sin\phi}{\sqrt{(\omega_0^2 - \omega^2)^2 + \omega^2\gamma^2}} \quad \text{since} \quad \sin\phi = \gamma\omega / \sqrt{(\omega_0^2 - \omega^2)^2 + \omega^2\gamma^2}$$

$$\left(\phi \equiv \arctan\left(\tfrac{\omega\gamma}{\omega_0^2 - \omega^2}\right)\right)$$

Quality Factor

$$m\ddot{x} = -\underbrace{m\omega_0^2 x}_{\text{spring}} - \underbrace{m\gamma\dot{x}}_{\text{friction}} + \underbrace{mF\cos\omega t}_{\text{forcing}}.$$

Energy content of transient motion that the CF describes

$$E = \tfrac{1}{2}(m\dot{x}^2 + m\omega_0^2 x^2) \qquad x = \mathrm{e}^{-\gamma t/2} A\cos(\omega_\gamma t + \psi)$$

$$= \tfrac{1}{2} m A^2 \mathrm{e}^{-\gamma t}[\tfrac{1}{4}\gamma^2\cos^2\eta + \omega_\gamma\gamma\cos\eta\sin\eta + \omega_\gamma^2\sin^2\eta + \omega_0^2\cos^2\eta] \quad (\eta \equiv \omega_\gamma t + \psi)$$

$$E \simeq \tfrac{1}{2} m(\omega_0 A)^2 \mathrm{e}^{-\gamma t} \qquad \text{small } \frac{\gamma}{\omega_0} \qquad (\omega_\gamma \equiv \omega_0\sqrt{1 - \tfrac{1}{4}\gamma^2/\omega_0^2} \simeq \omega_0)$$

Quality factor

$$Q \equiv \frac{E(t)}{E(t - \pi/\omega_0) - E(t + \pi/\omega_0)} \simeq \frac{1}{\mathrm{e}^{\pi\gamma/\omega_0} - \mathrm{e}^{-\pi\gamma/\omega_0}} = \tfrac{1}{2}\csc h(\pi\gamma/\omega_0)$$

$$\simeq \frac{\omega_0}{2\pi\gamma} \quad (\text{for small } \gamma/\omega_0).$$

Q is the inverse of the fraction of the oscillator's energy that is dissipated in one period

- approximately the number of oscillations before the energy decays by factor e