Second order linear equation with constant coefficients

The particular integral :

$$Lf = a_2 \frac{\mathrm{d}^2 f}{\mathrm{d}x^2} + a_1 \frac{\mathrm{d}f}{\mathrm{d}x} + a_0 f = h(x)$$

Sinusoidal h

$$Lf = a_2 \frac{d^2 f}{dx^2} + a_1 \frac{df}{dx} + a_0 f = h(x).$$

$$h = H\cos x \quad \text{so} \quad Lf \equiv a_2 f'' + a_1 f' + a_0 f = H\cos x$$

$$Lg(x) \equiv a_2 g'' + a_1 g' + a_0 g = He^{ix}$$

$$\Re e(Lg) = L[\Re e(g)] = \Re e(He^{ix}) = H\Re e(e^{ix}) = H\cos x$$

$\Rightarrow$ $f = \Re e(g)$ is the solution to the real equation

Solution: $g = Pe^{ix}$ $\quad Lg = (-a_2 + ia_1 + a_0)Pe^{ix} \quad \Rightarrow \quad P = \frac{H}{-a_2 + ia_1 + a_0}.$

$$f = \Re e\left(\frac{He^{ix}}{(a_0 - a_2) + ia_1}\right) = H\frac{(a_0 - a_2)\cos x + a_1 \sin x}{(a_0 - a_2)^2 + a_1^2}.$$

Ex 5

$$f'' + 3f' + 2f = \cos x.$$

$$g'' + 3g' + 2g = \mathrm{e}^{\mathrm{i}x}.$$

$$g = P\mathrm{e}^{\mathrm{i}x} \quad \text{where} \quad P = \frac{1}{-1 + 3\mathrm{i} + 2}.$$

PI

$$f_1 = \Re\mathrm{e}\left(\frac{\mathrm{e}^{\mathrm{i}x}}{1 + 3\mathrm{i}}\right) = \tfrac{1}{10}(\cos x + 3\sin x).$$

CF

$$f_0 = Ae^{-x} + Be^{-2x}$$

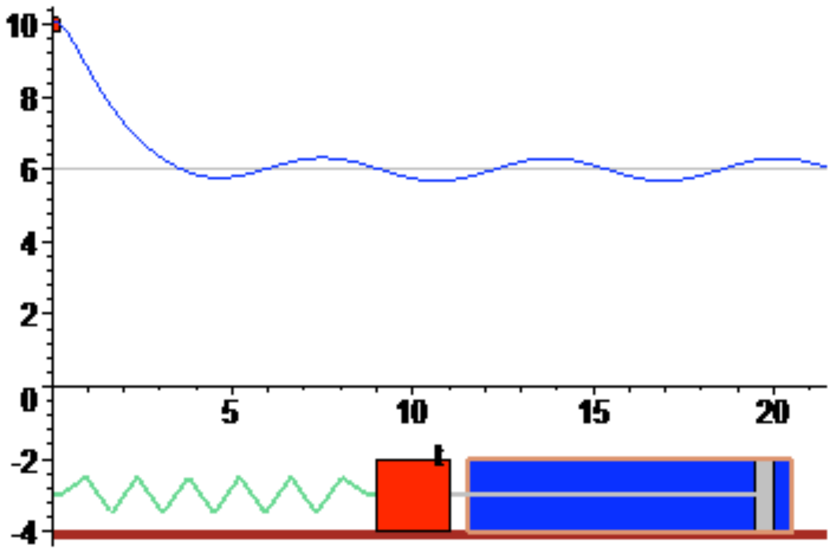
General solution

$$f = Ae^{-x} + Be^{-2x} + \tfrac{1}{10}(\cos x + 3\sin x)$$

$$f'' + 3f' + 2f = \cos x.$$

$$f = Ae^{-x} + Be^{-2x} + \tfrac{1}{10}(\cos x + 3\sin x)$$

$x(0) = 4, \quad x'(0) = 1 \quad \Rightarrow \quad A = \dfrac{17}{2} \quad B = -\dfrac{23}{5}$



Ex 5

$$f'' + 3f' + 2f = \cos x.$$

What if $\cos(x) \rightarrow \sin(x)$?

$$g'' + 3g' + 2g = e^{ix}.$$

$$f = \Im m(g)$$

$$g = Pe^{ix} \quad \text{where} \quad P = \frac{1}{-1 + 3i + 2}.$$

PI

$$f_1 = \Im m\left(\frac{e^{ix}}{1 + 3i}\right) = \tfrac{1}{10}(\sin x - 3\cos x).$$

CF

$$f_0 = Ae^{-mx} + Be^{-2mx}$$

General solution

$$f = Ae^{-mx} + Be^{-2mx} + \tfrac{1}{10}(\sin x - 3\cos x)$$

Ex 6 $$f'' + 3f' + 2f = 3\cos x + 4\sin x.$$

$$= 5\cos(x+\phi) = 5\Re\mathrm{e}(\mathrm{e}^{\mathrm{i}(x+\phi)})$$

where $\phi = \arctan(-4/3)$

Proof:

$$\cos(x+\phi) = \cos x \cos\phi - \sin x \sin\phi$$

$$A\cos x + B\sin x = \sqrt{A^2+B^2}\left(\frac{A}{\sqrt{A^2+B^2}}\cos x + \frac{B}{\sqrt{A^2+B^2}}\sin x\right)$$

$$= \sqrt{A^2+B^2}\cos(x+\phi),$$

$$\cos\phi = A/\sqrt{A^2+B^2}, \quad \sin\phi = -B/\sqrt{A^2+B^2}$$

and

$$\tan\phi = -B/A$$

Ex 6

$$f'' + 3f' + 2f = 3\cos x + 4\sin x.$$

$$= 5\cos(x+\phi) = 5\Re\text{e}(\text{e}^{\text{i}(x+\phi)})$$

where $\phi = \arctan(-4/3)$

$$g'' + 3g' + 2g = 5\text{e}^{\text{i}(x+\phi)}$$

Trial solution : $\text{g} = P\text{e}^{\text{i}(x+\phi)}$

$$P = \frac{5}{-1+3\text{i}+2} = \frac{5}{1+3\text{i}},$$

$$f_1 = 5\Re\text{e}\left(\frac{\text{e}^{\text{i}(x+\phi)}}{1+3\text{i}}\right) = \tfrac{1}{2}[\cos(x+\phi) + 3\sin(x+\phi)].$$

Ex 7

$$\boxed{f'' + f = \cos x} \Rightarrow \quad g'' + g = e^{ix}$$

C.F. $$\left(\frac{d}{dx} + i\right)\left(\frac{d}{dx} - i\right)g = e^{ix} \quad \Rightarrow \quad g = Ce^{ix}, \quad f = A\cos(x) + B\sin(x)$$

P.I. $$\left(\frac{d}{dx} + i\right)\left(\frac{d}{dx} - i\right)g = e^{ix}.$$

Try $g = Pxe^{ix}$

Then $$e^{ix} = \left(\frac{d}{dx} + i\right)\left(\frac{d}{dx} - i\right)Pxe^{ix} = \left(\frac{d}{dx} + i\right)Pe^{ix} = 2iPe^{ix}$$

$$\Rightarrow \quad P = \frac{1}{2i} \quad \Rightarrow \quad \boxed{f = \Re e\left(\frac{xe^{ix}}{2i}\right) = \tfrac{1}{2}x\sin x}$$

Ex 8

$$f'' + f = e^{-x}(3\cos x + 4\sin x) = 5\Re e(e^{-x} e^{i(x+\phi)})$$

where $\phi = \arctan(-4/3)$

Trial function

$$g = P e^{(i-1)x + i\phi}$$

$$P = \frac{5}{(i-1)^2 + 1} = \frac{5}{1-2i}.$$

PI $$f_1 = 5\Re e\left(\frac{e^{(i-1)x+i\phi}}{1-2i}\right) = e^{-x}[\cos(x+\phi) - 2\sin(x+\phi)].$$

Solutions with combinations of driving functions

$$Lf = a_2 \frac{\mathrm{d}^2 f}{\mathrm{d}x^2} + a_1 \frac{\mathrm{d}f}{\mathrm{d}x} + a_0 f = h_1(x) + h_2(x)$$

$$Lf_1 = a_2 \frac{\mathrm{d}^2 f_1}{\mathrm{d}x^2} + a_1 \frac{\mathrm{d}f_1}{\mathrm{d}x} + a_0 f_1 = h_1(x) \qquad Lf_2 = a_2 \frac{\mathrm{d}^2 f_2}{\mathrm{d}x^2} + a_1 \frac{\mathrm{d}f_2}{\mathrm{d}x} + a_0 f_2 = h_2(x)$$

Since the equation is linear a solution to the original equation is given by

$$f = f_1 + f_2$$