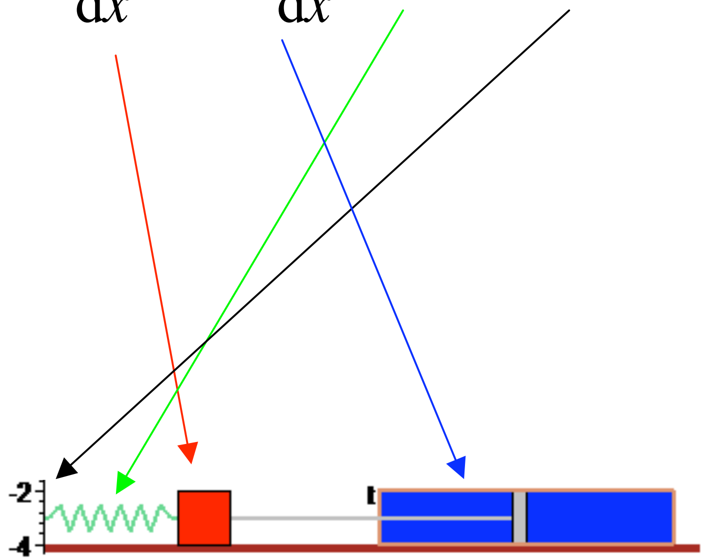
Second order linear equation with constant coefficients

$$Lf = a_2 \frac{\mathrm{d}^2 f}{\mathrm{d}x^2} + a_1 \frac{\mathrm{d}f}{\mathrm{d}x} + a_0 f = h(x).$$



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Solution

$$Lf = a_2 \frac{d^2 f}{dx^2} + a_1 \frac{df}{dx} + a_0 f = h(x).$$

The number of independent complementary functions is the number of integration constants – equal to the order of the differential equation

Complementary function

1) Construct f_0 the general solution to the homogeneous equation $Lf_0 = 0$

2) Find a solution, f_1, to the inhomogeneous equation $Lf_1 = h$

Particular integral

General solution : $f_0 + f_1$

For a nth order differential equation need n independent solutions to Lf=0 to specify the complementary function

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Complementary function

$$Lf = a_2 \frac{\mathrm{d}^2 f}{\mathrm{d}x^2} + a_1 \frac{\mathrm{d}f}{\mathrm{d}x} + a_0 f = 0.$$

Try $y = \mathrm{e}^{mx}$ ⇒ $a_2 m^2 + a_1 m + a_0 = 0.$

$$m_\pm \equiv \frac{-a_1 \pm \sqrt{a_1^2 - 4a_2 a_0}}{2a_2},$$

"Auxiliary" equation

$a_1^2 - 4a_2 a_0 \to +, 0, -$

⇒ Complementary function $y = A_+ \mathrm{e}^{m_+ x} + A_- \mathrm{e}^{m_- x}.$

Two constants of integration

Ex 1

$$\frac{d^2y}{dx^2}+4\frac{dy}{dx}+3y=0.$$

Auxiliary eq. $(m+3)(m+1)=0 \quad \Rightarrow \quad$ CF is $y = Ae^{-3x} + Be^{-x}$

.

If y(0)=2, y'(0)=0 $\Rightarrow \quad A+B=2, \quad -3A-B=0$

Initial conditions

A=-1, B=3 $\quad \Rightarrow \quad y=-e^{-3x}+3e^{-x}$

Ex 2

$$Ly = \frac{d^2 y}{dx^2} + 2\frac{dy}{dx} + 5y = 0.$$

Complex?

Auxiliary eq. $m^2 + 2m + 5 = 0$

i.e. $m = \frac{1}{2}(-2 \pm \sqrt{4-20}) = -1 \pm 2i \quad \Rightarrow$ CF is $y = Ae^{(-1+2i)x} + Be^{(-1-2i)x}$

But L is a real operator $\Rightarrow \quad 0 = \Re e(Ly) = L[\Re e(y)]$

i.e. $\Re e(y)$ is a solution (as is $\Im m(y)$) $\Rightarrow$ $\Re e(y) = e^{-x}[A' \cos(2x) + B' \sin(2x)].$

Find the solution for which $y(0) = 5$ and $(dy/dx)_0 = 0$

Initial conditions

$$5 = A'$$

$$0 = -A' + 2B' \quad \Rightarrow \quad B' = \tfrac{5}{2} \quad \Rightarrow \quad y = e^{-x}[5\cos(2x) + \tfrac{5}{2}\sin(2x)].$$

Factorisation of operators

We wish to solve $$Lf = a_2 \frac{d^2 f}{dx^2} + a_1 \frac{df}{dx} + a_0 f = 0.$$

We did this by trying $y = e^{mx}$

$$\Rightarrow \quad a_2 m^2 + a_1 m + a_0 = a_2 (m - m_+)(m - m_-) = 0. \qquad m_\pm = \frac{-a_1 \pm \sqrt{a_1^2 - 4a_2 a_0}}{2a_2}$$

This is equivalent to factorising the equation

$$\left(\frac{d}{dx} - m_-\right)\left(\frac{d}{dx} - m_+\right) f = \frac{d^2 f}{dx^2} - (m_- + m_+)\frac{df}{dx} + m_- m_+ f$$
$$= \frac{d^2 f}{dx^2} + \frac{a_1}{a_2}\frac{df}{dx} + \frac{a_0}{a_2} \equiv \frac{Lf}{a_2}$$

Now we can see why the CF is made up of exponentials …. because :

$$\left(\frac{d}{dx} - m_-\right) e^{m_- x} = 0 \quad ; \quad \left(\frac{d}{dx} - m_+\right) e^{m_+ x} = 0.$$

Factorisation of operators and repeated roots

$$Lf = a_2 \frac{d^2 f}{dx^2} + a_1 \frac{df}{dx} + a_0 f = 0.$$

$$(\frac{d}{dx} - m_-)(\frac{d}{dx} - m_+) f = 0 \qquad m_\pm = \frac{-a_1 \pm \sqrt{a_1{}^2 - 4a_2 a_0}}{2a_2}$$

What happens if $a_1^2 - 4a_2 a_0 = 0$ and $m_+ = m_- = m$?

$$Lf = (\frac{d}{dx} - m)(\frac{d}{dx} - m) f.$$

e^{mx} gives one solution : $L(e^{mx}) = (\frac{d}{dx} - m)(\frac{d}{dx} - m) e^{mx} = 0$

xe^{mx} gives the second independent solution :

$$L(xe^{mx}) = (\frac{d}{dx} - m)(\frac{d}{dx} - m) xe^{mx} = (\frac{d}{dx} - m) e^{mx} = 0,$$

$$y = Ae^{mx} + Bxe^{mx}$$

Ex 4 $$\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + y = 0.$$

Auxiliary equation $(m+1)^2 = 0$

$$\Rightarrow \quad \boxed{y = Ae^{-x} + Bxe^{-x}}$$

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Ex 5 $$\frac{d^4y}{dx^4} - 2\frac{d^3y}{dx^3} + 2\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + y = 0.$$

Auxiliary equation $(m-1)^2(m-i)(m+i) = 0$

$$\Rightarrow \quad \boxed{y = e^x(A + Bx) + C\cos x + D\sin x.}$$