

Almost separable equations

$$\frac{dy}{dx} = f(ax + by)$$

Change variables : $z = ax + by$ $\frac{dz}{dx} = a + b \frac{dy}{dx}$

$$\frac{dz}{dx} = a + bf(z) \Rightarrow x = \int \frac{1}{(a + bf(z))} dz.$$

Ex 2

$$\frac{dy}{dx} = (-4x + y)^2$$

$$z = y - 4x \Rightarrow \frac{dz}{dx} = -4 + \frac{dy}{dx} = z^2 - 4$$

$$x = \frac{1}{4} \ln\left(\frac{z-2}{z+2}\right) + C$$

$$\Rightarrow y = 4x + 2 \frac{(1+ke^{4x})}{(1-ke^{4x})}$$

← k a constant

Homogeneous equations

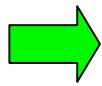
$$\frac{dy}{dx} = f(y/x).$$

The equation is invariant under $x \rightarrow sx$, $y \rightarrow sy$ **homogeneous**

Solution

$$y = vx \quad \Rightarrow \quad y' = v'x + v.$$

$$i.e. \quad v' = \frac{1}{x}(f(v) - v)$$



$$\int \frac{dv}{f(v)-v} = \int \frac{dx}{x} = \ln x + \text{constant.}$$

Ex 3

$$xy \frac{dy}{dx} - y^2 = (x + y)^2 e^{-y/x}$$

Homogeneous

Change variables $y = vx \Rightarrow y' = v'x + v$.

$$(v'x + v) - v = \frac{(1+v)^2}{v} e^{-v} \Rightarrow \ln x = \int \frac{e^v v dv}{(1+v)^2}.$$

To evaluate integral change variables $u \equiv 1 + v$

$$e^{-1} \int \left(\frac{1}{u} - \frac{1}{u^2} \right) e^u du = e^{-1} \left[\frac{e^u}{u} \right].$$

$$i.e. \quad \ln x = \frac{e^{\frac{y}{x}}}{1 + \frac{y}{x}}$$

Homogeneous but for constants

$$\frac{dy}{dx} = \frac{x + 2y + 1}{x + y + 2}$$

$$x = x' + a, \quad y = y' + b \quad \Rightarrow \quad \frac{dy}{dx} = \frac{dy'}{dx'} = \frac{dy'}{dx'} \cdot \frac{dx'}{dx} = \frac{dy'}{dx'}$$

$$\frac{dy'}{dx'} = \frac{x' + 2y' + 1 + a + 2b}{x' + y' + 2 + a + b}$$

$1 + a + 2b = 0$

$2 + a + b = 0$

$a = -3, \quad b = 1$

$$\frac{dy'}{dx'} = \frac{x' + 2y'}{x' + y'}$$

Homogeneous

The Bernoulli equation

$$\frac{dy}{dx} + P(x)y = Q(x)y^n, \quad n \neq 1$$

To solve, change variable to $z = y^{1-n} \Rightarrow \frac{dz}{dx} = (1-n)y^{-n} \frac{dy}{dx}$

Gives the equation

$$\frac{dz}{dx} + (1-n)P(x)z = (1-n)Q(x)$$

1st order Linear

Ex 4

$$y' + y = y^{2/3}$$

$$z = y^{1-n} = y^{1/3} \Rightarrow z' + \frac{z}{3} = \frac{1}{3}$$

1st order linear

Integrating factor $e^{x/3} \Rightarrow ze^{x/3} = \int e^{x/3} dx / 3$

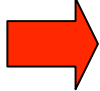
$$z = y^{1/3} = 1 + ce^{-x/3}$$

Second order linear equations

General form :

$$\frac{d^2 f}{dx^2} + p(x) \frac{df}{dx} + q(x) f = h(x).$$

Integrating factor? Suppose $\exists I(x)$ such that $\frac{d^2 If}{dx^2} = Ih$

 $2 \frac{dI}{dx} = Ip$ and $\frac{d^2 I}{dx^2} = Iq.$

These equations are incompatible in most cases....

We will study a subset of 2nd order equations which appear in a wide variety of physical processes and which can be analytically solved....

Const coefficients : $p(x)=\text{const.}, q(x)=\text{const}$