# Almost separable equations

$$\frac{dy}{dx} = f(ax + by)$$

Change variables : $z = ax + by$ $\qquad \frac{dz}{dx} = a + b\frac{dy}{dx}$

$$\frac{dz}{dx} = a + bf(z) \quad \Rightarrow \quad \boxed{x = \int \frac{1}{(a + bf(z))} dz.}$$

Ex 2 $\quad \boxed{\frac{dy}{dx} = (-4x + y)^2}$ $\qquad z = y - 4x \qquad \Rightarrow \quad \frac{dz}{dx} = -4 + \frac{dy}{dx} = z^2 - 4$

$$x = \tfrac{1}{4}\ln\left(\tfrac{z-2}{z+2}\right) + C \qquad \boxed{\Rightarrow \quad y = 4x + 2\tfrac{(1+ke^{4x})}{(1-ke^{4x})}}$$

k a constant

## Homogeneous equations

$$\frac{dy}{dx} = f(y/x).$$

The equation is invariant under $x \to sx, \quad y \to sy$ .... homogeneous

Solution $$y = \mathrm{v}x \quad \Rightarrow \quad y' = \mathrm{v}'x + \mathrm{v}.$$

$$i.e. \quad \mathrm{v}' = \frac{1}{x}(f(\mathrm{v}) - \mathrm{v})$$

$$\Rightarrow \quad \int \frac{dv}{f(\mathrm{v}) - \mathrm{v}} = \int \frac{dx}{x} = \ln x + \text{constant}.$$

Ex 3

$$xy\frac{\mathrm{d}y}{\mathrm{d}x} - y^2 = (x+y)^2 \mathrm{e}^{-y/x}$$

Homogeneous

Change variables $y = \mathrm{v}x \quad \Rightarrow \quad y' = \mathrm{v}'x + \mathrm{v}.$

$$(\mathrm{v}'x + \mathrm{v}) - \mathrm{v} = \frac{(1+\mathrm{v})^2}{\mathrm{v}} \mathrm{e}^{-\mathrm{v}} \quad \Rightarrow \quad \ln x = \int \frac{\mathrm{e}^{\mathrm{v}} \mathrm{v} \mathrm{d}\mathrm{v}}{(1+\mathrm{v})^2}.$$

To evaluate integral change variables $u \equiv 1 + \mathrm{v}$

$$\mathrm{e}^{-1} \int \left(\frac{1}{u} - \frac{1}{u^2}\right) \mathrm{e}^u \mathrm{d}u = \mathrm{e}^{-1} \left[\frac{\mathrm{e}^u}{u}\right].$$

$$i.e. \quad \ln x = \frac{e^{\frac{y}{x}}}{1 + \frac{y}{x}}$$

## Homogeneous but for constants

$$\frac{dy}{dx} = \frac{x + 2y + 1}{x + y + 2}$$

$$x = x' + a, \quad y = y' + b \qquad \Rightarrow \quad \frac{dy}{dx} = \frac{dy'}{dx} = \frac{dy'}{dx'} . \frac{dx'}{dx} = \frac{dy'}{dx'}$$

$$1 + a + 2b = 0$$

$$\frac{dy'}{dx'} = \frac{x' + 2y' + 1 + a + 2b}{x' + y' + 2 + a + b}$$

$$a = -3, \quad b = 1$$

$$2 + a + b = 0$$

$$\frac{dy'}{dx'} = \frac{x' + 2y'}{x' + y'}$$

Homogeneous

## The Bernoulli equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} + P(x)y = Q(x)y^n, \qquad n \neq 1$$

To solve, change variable to $z = y^{1-n} \quad \Rightarrow \quad \frac{dz}{dx} = (1-n)y^{-n}\frac{dy}{dx}$

Gives the equation

$$\boxed{\frac{dz}{dx} + (1-n)P(x)z = (1-n)Q(x)}$$

1st order Linear

Ex 4 $\boxed{y' + y = y^{2/3}}$ $\quad z = y^{1-n} = y^{1/3} \quad \Rightarrow \quad z' + \frac{z}{3} = \frac{1}{3}$

1st order linear

Integrating factor $\mathrm{e}^{x/3} \quad \Rightarrow \quad ze^{x/3} = \int e^{x/3}dx/3$

$$\boxed{z = y^{1/3} = 1 + ce^{-x/3}}$$

## Second order linear equations

General form :

$$\frac{\mathrm{d}^2 f}{\mathrm{d}x^2} + p(x)\frac{\mathrm{d}f}{\mathrm{d}x} + q(x)f = h(x).$$

Integrating factor? Suppose $\exists \; I(x)$ such that $\frac{\mathrm{d}^2 If}{\mathrm{d}x^2} = Ih$

$$2\frac{\mathrm{d}I}{\mathrm{d}x} = Ip \quad \text{and} \quad \frac{\mathrm{d}^2 I}{\mathrm{d}x^2} = Iq.$$

These equations are incompatible in most cases….

We will study a subset of $2^{nd}$ order equations which appear in a wide variety of physical processes and which can be analytically solved….

Const coefficients : p(x)=const., q(x)=const