Differential Equations

e.g. $$\frac{\mathrm{d}f}{\mathrm{d}x} + xf = \sin x$$ + Initial conditions

Differential operators

Functions : map numbers → numbers $x \to e^x$

Operators : map functions → functions $f \to \alpha f; f \to 1/f; f \to f + \alpha$

Differential operators : $f \to \frac{\mathrm{d}f}{\mathrm{d}x}; f \to \frac{\mathrm{d}^2 f}{\mathrm{d}x^2}; f \to 2\frac{\mathrm{d}^2 f}{\mathrm{d}x^2} + f\frac{\mathrm{d}f}{\mathrm{d}x}; \ldots)$

Convenient to name operator $e.g.\quad L(f) \;:\; f \to \frac{\mathrm{d}f}{\mathrm{d}x}$

Order of a differential operator

$$L_1(f) \equiv \frac{df}{dx} + 3f \quad \text{is first order,}$$

$$L_2(f) \equiv \frac{d^2 f}{dx^2} + 3f \quad \text{is second order,}$$

$$L_3(f) \equiv \frac{d^2 f}{dx^2} + 4\frac{df}{dx} \quad \text{is second order.}$$

Linear operator

α, β real or complex numbers

$$\text{If } L(\alpha f + \beta g) = \alpha L(f) + \beta L(g),$$

then L is a linear operator

e.g. $f \to \frac{df}{dx}$ and $f \to \alpha f$ are linear

$f \to \frac{1}{f}$ and $f \to f + \alpha$ are not linear

The principle of superposition

Suppose f and g are solutions to $L(y) = 0$ for different initial conditions

$$i.e. \quad L(f) = 0, \quad L(g) = 0$$

Consider the Linear Combination : $\alpha f + \beta g$

If L linear then $\mathrm{L}(\alpha f + \beta g) = \alpha L(f) + \beta L(g) = 0$

i.e. a linear combination of solutions of a linear operator is also a solution –

“principle of superposition”

Inhomogeneous terms

$$L(f) = h(x)$$

e.g. $$\frac{\mathrm{d}f}{\mathrm{d}x} + xf = \sin x$$

Sometimes called “driving” term

Solution to differential equations

The number of independent complementary functions is the number of integration constants – equal to the order of the differential equation

$$L(f) = h(x)$$

Complementary function

1) Construct f_0 the general solution to the homogeneous equation $Lf_0 = 0$

2) Find a solution, f_1, to the inhomogeneous equation $Lf_1 = h$

Particular integral

General solution : $f_0 + f_1$

For a nth order differential equation need n independent solutions to Lf=0 to specify the complementary function

First order linear equations

General form : $\frac{\mathrm{d}f}{\mathrm{d}x}+q(x)f=h(x).$

Easy to solve

Integrating factor

Look for a function I(x) such that $\mathrm{I(x)}\frac{\mathrm{d}f}{\mathrm{d}x}+I(x)q(x)f\equiv\frac{dIf}{dx}=I(x)h(x)$

Solution: CF $\frac{dIf}{dx}=0$ is $If=const$

PI $\mathrm{I(x)}f(x)=\int^{x}I(x')h(x')dx'$

$$\text{Solution} \; : f(x)=\frac{1}{I(x)}\int_{x_0}^{x}I(x')h(x')dx'$$

First order linear equations

$$\text{General form}: \frac{\mathrm{d}f}{\mathrm{d}x} + q(x)f = h(x).$$

Easy to solve

Integrating factor

Look for a function I(x) such that $\mathrm{I}(\mathrm{x})\frac{\mathrm{d}f}{\mathrm{d}x} + I(x)q(x)f \equiv \frac{dIf}{dx} = I(x)h(x)$

$$\text{Solution}: f(x) = \frac{1}{I(x)}\int_{x_0}^{x} I(x')h(x')dx'$$

First order :
1 integration constant (CF)

We have $I(x)q(x) = \frac{dI}{dx}$

$$\Rightarrow \ln(I(x)) = \int^{x} q(x')dx' \quad \Rightarrow \quad I(x) = e^{\int^{x} q(x')dx'}$$

"Integrating factor"

Ex1 Solve

$$2x\frac{\mathrm{d}f}{\mathrm{d}x} - f = x^2.$$

Writing in “standard” form :

$$\frac{\mathrm{d}f}{\mathrm{d}x} - \frac{f}{2x} = \tfrac{1}{2}x$$

$$\text{so } q = -\frac{1}{2x} \text{ and } I = \mathrm{e}^{-\frac{1}{2}\ln x} = \frac{1}{\sqrt{x}}.$$

Plugging this into the form of the solution we have :

$$\boxed{f = \tfrac{1}{2}\sqrt{x}\int_{x_0}^{x}\sqrt{x'}\,\mathrm{d}x' = \tfrac{1}{3}\left(x^2 - x_0^{3/2}x^{1/2}\right)}$$

First order nonlinear equations

Although no general method for solution is available, there are several cases of physically relevant nonlinear equations which can be solved analytically :

Separable equations

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{f(x)}{g(y)}$$

Solution :

$$\int g(y)dy = \int f(x)dx$$

Ex 1

$$\frac{\mathrm{d}y}{\mathrm{d}x} = y^2 \mathrm{e}^x \quad \Rightarrow \quad \int \frac{dy}{y^2} = \int \mathrm{e}^x \mathrm{d}x$$

i.e $\frac{-1}{y} = \mathrm{e}^x + c$ or

$$y = \frac{-1}{(\mathrm{e}^x + c)}$$