# Complex Numbers II

**1**. Write the following in the form $a + \mathrm{i}b$, where $a$ and $b$ are real:

(i) $\mathrm{e}^{\mathrm{i}}$, (ii) $\sqrt{\mathrm{i}}$, (iii) $\ln \mathrm{i}$, (iv) $\cos \mathrm{i}$, (v) $\sin \mathrm{i}$,

(vi) $\sinh(x + \mathrm{i}y)$, (vi) $ln\frac{1}{2}(\sqrt{3}+i)$, (vii) $(1+i)^{iy}$.

(Note $i = e^{i\pi/2}$)

**2**. Sketch the curves $C_1$ and $C_2$ in the Argand diagram for $z$ defined respectively by $\arg[(z-4)/(z-1)] = \pi/2$ and $\arg[(z-4)/(z-1)] = 3\pi/2$.

**3**. Prove that

$$\sum_{r=1}^{n} \binom{n}{r} \sin 2r\theta = 2^n \sin n\theta \cos^n \theta \quad \text{where} \quad \binom{n}{r} \equiv \frac{n!}{(n-r)!r!}.$$

[Hint: express the left side as $Im\left(\sum \binom{n}{r} \mathrm{e}^{\mathrm{i}2r\theta}\right)$ and use the binomial expansion.]

**4**. Show that

$$\sum_{n=0}^{\infty} 2^{-n} \cos n\theta = \frac{1 - \frac{1}{2}\cos\theta}{\frac{5}{4} - \cos\theta}.$$

**5**. Find the 5th roots of unity and plot them on an Argand diagram. What is the sum of the roots?

**6**. Solve the equation $z^4 = -4i$.

**7**. Show that the equation with the four roots $z = \frac{1}{2}(\pm\sqrt{3} \pm i)$ is $z^4 - z^2 + 1 = 0$.

**8**. Show that the equation $(z + \mathrm{i})^n - (z - \mathrm{i})^n = 0$ has roots $z = \cot(r\pi/n)$, where $r = 1, 2, \ldots, n-1$.

**9**. Find the roots of the equation $(z-1)^n + (z+1)^n = 0$. Hence solve the equation $x^3 + 15x^2 + 15x + 1 = 0$.

**10**. Prove that the sum and product of the roots, $x_i$, of the polynomial $a_n x^n + \cdots + a_0$ satisfy $\sum x_i = -a_{n-1}/a_n$ and $\prod x_i = (-1)^n a_0/a_n$. Hence find the sum and the product of the roots of $P = x^3 - 6x^2 + 11x - 6$. Show that $x = 1$ is a root and by writing $P = (x-1)Q$, where $Q$ is a quadratic, find the other two roots. Verify that the roots have the expected sum and product.

**11**. Show that the equation $(z+1)^n - \mathrm{e}^{2in\theta}(z-1)^n = 0$ has root $z = -\mathrm{i}\cot(\theta + r\pi/n)$. Show that

$$\prod_{r=1}^{n} \cot\left(\theta + \frac{r\pi}{n}\right) = \begin{cases} (-1)^{n/2} & \text{for } n \text{ even} \\ (-1)^{(n-1)/2} \cot n\theta & \text{for } n \text{ odd.} \end{cases}$$

**12**. Find all the roots, real and complex, of the equation $z^3 - 1 = 0$. If $\omega$ is one of the complex roots, prove that $1 + \omega + \omega^2 = 0$. Find the sums of the following series:

$$S_1 = 1 + \frac{x^3}{3!} + \frac{x^6}{6!} + \cdots; \quad S_2 = x + \frac{x^4}{4!} + \frac{x^7}{7!} + \cdots; \quad S_3 = \frac{x^2}{2!} + \frac{x^5}{5!} + \frac{x^8}{8!} + \cdots$$

[Hint: note that $S_1 + S_2 + S_3 = \mathrm{e}^x$ and calculate $\mathrm{e}^{\omega x}$ and $\mathrm{e}^{\omega^2 x}$.]