Roots of polynomials

$$P(z) \equiv a_n z^n + a_{n-1} z^{n-1} + \cdots + a_0 .$$

$$P(z = z_i) = 0 \qquad \Rightarrow z_i \text{ is a root}$$

Characterising a polynomial by its roots ..the "fundamental theorem of algebra"

$$a_n z^n + a_{n-1} z^{n-1} + \cdots + a_0 = a_n (z - z_1)(z - z_2) \cdots (z - z_n)$$

In mathematics, the fundamental theorem of algebra states that every non-zero single-variable polynomial, with complex coefficients, has exactly as many complex roots as its degree, if repeated roots are counted up to their multiplicity.

Roots of polynomials

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Characterising a polynomial by its roots

$$a_n z^n + a_{n-1} z^{n-1} + \cdots + a_0 = a_n (z - z_1)(z - z_2) \cdots (z - z_n)$$

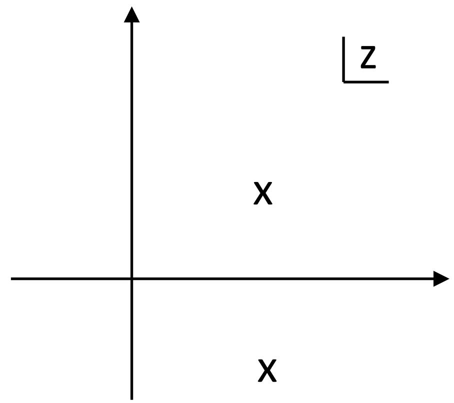
$$= a_n \left(z^n - z^{n-1} \sum_{j=1}^{n} z_j + \cdots + (-1)^n \prod_{j=1}^{n} z_j \right).$$

Comparing coefficients of z^{n-1} and z^0

$$\boxed{\frac{a_{n-1}}{a_n} = -\sum_j z_j \quad ; \quad \frac{a_0}{a_n} = (-1)^n \prod_j z_j}$$

e.g. quadratic equations : $a_2 x^2 + a_1 x + a_0$

Roots: $$x_{1,2} = \frac{\left(-a_1 \pm \sqrt{a_1^2 - 4a_2 a_0}\right)}{2a_2}$$



If complex, roots come in complex conjugate pairs

Sum of roots $\frac{a_1}{a_2} = -(x_1 + x_2)$ Product of roots $\frac{a_0}{a_2} = x_1 . x_2$

for roots

- General solutions not available for higher order polynomials (quartics and above)

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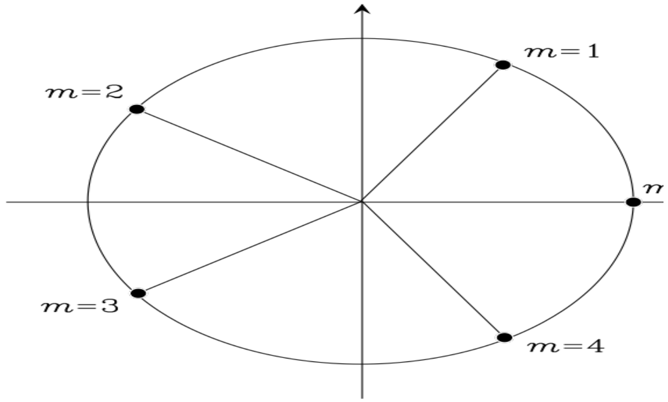
Can find solutions in special cases....

Example 1 : nth roots of unity :

$$x^n = 1 \quad (i.e.\, x^n - 1 = 0)$$



$$\Rightarrow \quad x = 1^{1/n}$$

$$1 = e^{2m\pi\imath} \quad \Rightarrow \quad 1^{1/n} = e^{2m\pi\imath/n}$$

$$= \cos(\frac{2m\pi}{n}) + \imath \sin(\frac{2m\pi}{n})$$

$$1^{1/5} = \cos(\frac{2m\pi}{5}) + \imath \sin(\frac{2m\pi}{5}) \quad (m = 0,1,2,3,4).$$

Example 2 : Roots of polynomials

$$(z+\imath)^7+(z-\imath)^7=0$$

$$\left(\frac{z+\imath}{z-\imath}\right)^7=-1=\mathrm{e}^{(2m+1)\pi\imath}$$

$$\Rightarrow\quad \frac{z+\imath}{z-\imath}=\mathrm{e}^{(2m+1)\pi\imath/7}$$

$$\Rightarrow\quad z(1-\mathrm{e}^{(2m+1)\pi\imath/7})=-\imath(1+\mathrm{e}^{(2m+1)\pi\imath/7})$$

$$\Rightarrow\quad z=\imath\frac{\mathrm{e}^{(2m+1)\pi\imath/7}+1}{\mathrm{e}^{(2m+1)\pi\imath/7}-1}$$

$$=\imath\frac{\mathrm{e}^{(2m+1)\pi\imath/14}+\mathrm{e}^{-(2m+1)\pi\imath/14}}{\mathrm{e}^{(2m+1)\pi\imath/14}-\mathrm{e}^{-(2m+1)\pi\imath/14}}=\imath\frac{2\cos(\frac{2m+1}{14}\pi)}{2\imath\sin(\frac{2m+1}{14}\pi)}=\cot(\tfrac{2m+1}{14}\pi)$$

m=0,1,2,3,4,5,6

Example 2 : an alternative form

$$(z+\imath)^7 + (z-\imath)^7 = 0$$

We will often need the coefficient of $x^r y^{n-r}$ in $(x+y)^n$

These are conveniently obtained from Pascal's triangle :

$(x+y)^0$						1					
$(x+y)^1$					1		1				
$(x+y)^2$				1		2		1			
$(x+y)^3$			1		3		3		1		
$(x+y)^4$		1		4		6		4		1	
$(x+y)^5$	1		5		10		10		5		1

Solution:

$z = \cot(\frac{2m+1}{14}\pi)$

7th row of Pascal's triangle is 1 7 21 35 35 21 7 1 so

$$(z+\imath)^7 + (z-\imath)^7 = 0 \quad \Rightarrow \quad z^7 - 21z^5 + 35z^3 - 7z = 0$$

Example 2 : yet another form

The original equation $(z+\imath)^7+(z-\imath)^7=0 \quad \Rightarrow \quad z^7-21z^5+35z^3-7z=0$

can be written in another form :

$$z^7-21z^5+35z^3-7z=0$$

$$\Rightarrow \quad z^6-21z^4+35z^2-7=0 \quad or \quad z=0$$

$z_m, \quad m=3$

$$\Rightarrow \quad w^3-21w^2+35w-7=0 \quad (w\equiv z^2)$$

Hence the roots of $w^3-21w^2+35w-7=0$ are

$$w=\cot^2(\tfrac{2m+1}{14}\pi) \qquad (m=0,1,2)$$

Sum of roots $\Rightarrow \sum_{m=0}^{2}\cot^2(\tfrac{2m+1}{14}\pi)=21$

Ex 3 Another example where the underlying equation is not obvious :

$$\boxed{z^3 + 7z^2 + 7z + 1 = 0.}$$

$(x+y)^0$						1					
$(x+y)^1$					1		1				
$(x+y)^2$				1		2		1			
$(x+y)^3$			1		3		3		1		
$(x+y)^4$		1		4		6		4		1	
$(x+y)^5$	1		5		10		10		5		1

9th row of Pascal's triangle is $1 \;\; 8 \;\; 28 \;\; 56 \;\; 70 \;\; 56 \;\; 28 \;\; 8 \;\; 1$ so

$$\tfrac{1}{2}[(z+1)^8 - (z-1)^8] = 8z^7 + 56z^5 + 56z^3 + 8z$$

$$= 8z[w^3 + 7w^2 + 7w + 1] \quad (w \equiv z^2).$$

Now $(z+1)^8 - (z-1)^8 = 0$ when $\frac{z+1}{z-1} = e^{2m\pi i/8}$

i.e. when $z = \dfrac{e^{m\pi\imath/4} + 1}{e^{m\pi\imath/4} - 1} = -\imath\cot(m\pi/8) \quad (m = 1, 2, \ldots, 7),$

so the roots of the given equation are

$$\boxed{z = -\cot^2(m\pi/8) \qquad m = 1, 2, 3}$$

Ex 4 Show that

$$\frac{z^{2m}-a^{2m}}{z^2-a^2}=(z^2-2az\cos\frac{\pi}{m}+a^2)(z^2-2az\cos\frac{2\pi}{m}+a^2)\cdots(z^2-2az\cos\frac{(m-1)\pi}{m}+a^2).$$

i.e. Show that $P(z)=Q(z)$ where

$$P(z)\equiv z^{2m}-a^{2m}$$

(Roots : $z_r = ae^{r\pi i/m}$)

$$Q(z)\equiv(z^2-a^2)(z^2-2az\cos\frac{\pi}{m}+a^2)(z^2-2az\cos\frac{2\pi}{m}+a^2)\cdots(z^2-2az\cos\frac{(m-1)\pi}{m}+a^2).$$

Roots

$$z_r = a\cos\frac{r\pi}{m}\pm\sqrt{a^2\cos^2\frac{r\pi}{m}-a^2}$$

$$=a(\cos\frac{r\pi}{m}\pm\iota\sqrt{1-\cos^2\frac{r\pi}{m}})=ae^{\pm\iota r\pi/m} \quad (r=0,1,..,m).$$

Leading coefficient $a_{2m}=1$ ⇨ P(z) and Q(z) identical