de Moivre's theorem and trigonometric identities

$$z^n = (re^{\imath\theta})^n = r^n e^{in\theta} = r^n(\cos n\theta + i\sin n\theta)$$

For r=1

$$(e^{\imath\theta})^n = (\cos\theta + \imath\sin\theta)^n = \cos n\theta + \imath\sin n\theta.$$

De Moivre

e.g. n=2 :

$$\cos 2\theta = \cos^2\theta - \sin^2\theta$$

$$\sin 2\theta = 2\cos\theta\sin\theta$$

Uses of de Moivre and complex exponentials

Ex. 1 Find $(1+\imath)^8$

. Taking powers is much simpler in polar form so we write

$$(1+\imath) = \sqrt{(2)}\,\mathrm{e}^{\imath\pi/4}$$

. Hence

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$$(1+\imath)^8 = (\sqrt{(2)}\,\mathrm{e}^{\imath\pi/4})^8 = 16\mathrm{e}^{2\pi\imath} = 16$$

Uses of de Moivre and complex exponentials

Ex. 2 Solving differential equations

The solution to $\frac{d^2z}{d\theta^2} + z = 0$ is simply given by

$z = Ce^{i\theta}$ where $C = A + \imath B$ is a complex constant.

$z = x + i\,\mathrm{y}$

$$\mathrm{y} = \mathrm{Im}(z) = \mathrm{Im}((A + \imath B)(\cos\theta + \imath\sin\theta))$$
$$= A\sin\theta + B\cos\theta$$

c.f. $\frac{d^2y}{d\theta^2} + y \equiv \mathrm{Im}(\frac{d^2z}{d\theta^2} + z) = 0,\ \mathrm{v} = A\cos\theta + B\sin\theta$

Uses of de Moivre and complex exponentials

Ex. 3 Summing series

e.g. Show $$\sum_{n=0}^{\infty} r^n \sin(2n+1)\theta = \frac{(1+r)\sin\theta}{1-2r\cos 2\theta + r^2}, \qquad 0 < r < 1$$

$$\sum_{n=0}^{\infty} r^n \sin(2n+1)\theta = \operatorname{Im}\sum_n r^n(\mathrm{e}^{\imath(2n+1)\theta}) = \operatorname{Im}(\mathrm{e}^{\imath\theta}\sum_n (r\mathrm{e}^{2\imath\theta})^n)$$

$$= \operatorname{Im}(\mathrm{e}^{\imath\theta}\frac{1}{1-r\mathrm{e}^{2\imath\theta}})$$

$$= \operatorname{Im}(\frac{\mathrm{e}^{\imath\theta}(1-r\mathrm{e}^{-2\imath\theta})}{(1-r\mathrm{e}^{2\imath\theta})(1-r\mathrm{e}^{-2\imath\theta})})$$

$$= \frac{\sin\theta + r\sin\theta}{1-2r\cos 2\theta + r^2}$$

Curves in the complex plane

Ex 1 $|z| = 1$

$$z = re^{i\theta} \quad \Rightarrow \quad r = 1, \quad \text{any } \theta$$

or $(a)^2 + (b)^2 = 1, \qquad z = a + ib$

Circle centre (0,0), radius 1

$$|z - z_0| = 1$$

Circle centre z_0, radius 1

$$(a - a_0)^2 + (b - b_0)^2 = 1$$

Curves in the complex plane

Ex 2 $$\left| \frac{z-i}{z+i} \right| = 1$$

$$| z - i | = | z + i |$$

Distance from (0,1) = distance from (0,-1) ⇨ Real axis

or $(a)^2 + (b-1)^2 = (a)^2 + (b+1)^2$ $\qquad z = a + ib$

⇨ $b = 0, \quad a$ arbitrary

Curves in the complex plane

Ex 3 $\arg(\frac{z}{z+1}) = \frac{\pi}{4}$

i.e. $\arg(z) - \arg(z+1) = \frac{\pi}{4}$

Take tan of both sides :

$$\tan(A - B) = \frac{\tan A - \tan B}{1 + \tan A \tan B}$$

$$\Rightarrow \quad \frac{\frac{b}{a} - \frac{b}{a+1}}{1 + \frac{b}{a} \cdot \frac{b}{a+1}} = 1 = \frac{b(a+1) - ba}{a(a+1) + b^2}$$

$$z = a + ib$$

$$b = a(a+1) + b^2$$

$$(a + \tfrac{1}{2})^2 + (b - \tfrac{1}{2})^2 = \tfrac{1}{2}$$

…but BEWARE…not all of circle satisfies equation…

$$\arg\left(\frac{z}{z+1}\right) = \frac{\pi}{4}$$

$$\frac{z}{z+1} = \frac{z}{z+1} \cdot \frac{z^*+1}{z^*+1} = \frac{z(z^*+1)}{|z+1|^2} = \frac{(a+ib)(a+1-ib)}{|z+1|^2} = \frac{a(a+1)+b^2+ib}{|z+1|^2}$$

$$\Rightarrow \quad b > 0 \text{ since } \arg\left(\frac{z}{z+1}\right) \text{ positive}$$

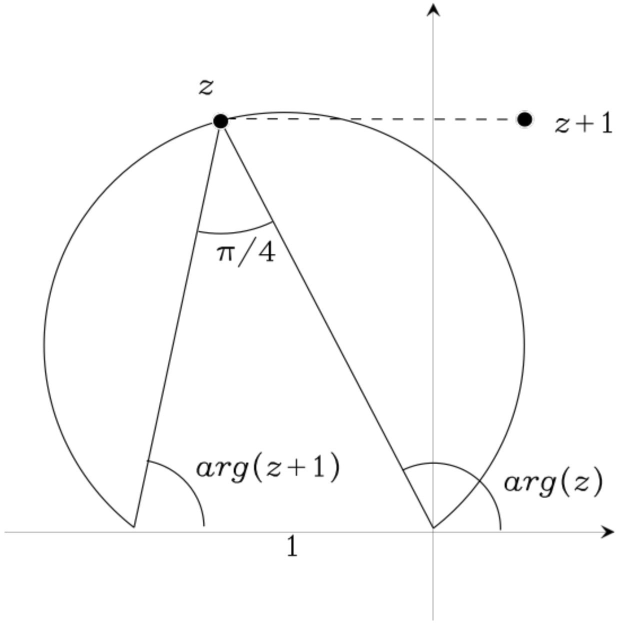
$$(a+\tfrac{1}{2})^2 + (b-\tfrac{1}{2})^2 = \tfrac{1}{2}, \quad b > 0$$

$\left(b < 0\right.$ solution introduced by tangent ambiguity$\left.\right)$

Alternative solution

$$\arg(\tfrac{z}{z+1}) = \tfrac{\pi}{4}$$

i.e. $\arg(z) - \arg(z+1) = \frac{\pi}{4}$



Solution : portion of circle through (0,0) and (-1,0)

Circle centre (-1/2,1/2) and radius $1/\sqrt{2}$

The lower portion of the circle is given by :

$$\arg\left(\frac{z}{z+1}\right) = -\frac{3\pi}{4}$$