Functions of complex numbers

- The complex exponential

Functions defined by power series :

$$e^x = 1 + x + \frac{x^2}{2!} + \ldots + \frac{x^n}{n!} + \ldots$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \ldots + (-1)^n \frac{x^{2n}}{(2n)!} + \ldots$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots + (-1)^n \frac{x^{2n+1}}{(2n+1)!} + \ldots$$

Define the complex exponential

$$e^\alpha = 1 + \alpha + \frac{\alpha^2}{2!} + \ldots + \frac{\alpha^n}{n!} + \ldots$$

$\alpha = a + ib$

Special case $\alpha = i\theta$

$$e^{i\theta} = \left(1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \ldots\right) + i\left(\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \ldots\right)$$

$$= \cos\theta + i\sin\theta$$

$$e^{i\theta} = (1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \ldots) + i\,(\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \ldots)$$

$$= \cos\theta + i\sin\theta$$

Used in :

$$z = |z|(\cos\theta + \imath\sin\theta) = |z|\,\mathrm{e}^{\imath\theta} \equiv re^{i\theta}$$

$$z^* = |z|(\cos\theta - \imath\sin\theta) = |z|\,\mathrm{e}^{-\imath\theta} \equiv re^{-\imath\theta}$$

$$\frac{1}{z} = \frac{z^*}{zz^*} = \frac{\mathrm{e}^{-\imath\theta}}{|z|} \equiv \frac{\mathrm{e}^{-\imath\theta}}{r}$$

$$e^{i\theta} = (1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \ldots) + i\,(\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \ldots)$$

$$= \cos\theta + i\sin\theta$$

Used in :

$$z = |z|(\cos\theta + \imath\sin\theta) = |z|\,\mathrm{e}^{\imath\theta} \equiv re^{i\theta}$$

Multiplication

$$z_1\, z_2 = r_1 e^{i\theta_1}\; r_2 e^{i\theta_2} = r_1\, r_2\, e^{i\theta_1} e^{i\theta_2} = r_1\, r_2\, e^{i(\theta_1+\theta_2)} = r_1\, r_2\left(\cos(\theta_1+\theta_2) + i\sin(\theta_1+\theta_2)\right)$$

Division

$$\frac{z_1}{z_2} = \frac{r_1 e^{i\theta_1}}{r_2 e^{i\theta_2}} = \frac{r_1}{r_2}\;\frac{e^{i\theta_1}}{e^{i\theta_2}} = \frac{r_1}{r_2}\, e^{i\theta_1} e^{-i\theta_2} = \frac{r_1}{r_2}\, e^{i(\theta_1-\theta_2)} = \frac{r_1}{r_2}\left(\cos(\theta_1-\theta_2) + i\sin(\theta_1-\theta_2)\right)$$

$$e^{i\theta} = (1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - ...) + i\,(\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - ...)$$

$$= \cos\theta + i\sin\theta$$

Can invert the relation :

$$\cos\theta = \frac{1}{2}(e^{i\theta} + e^{-i\theta})$$

$$\sin\theta = \frac{1}{2i}(e^{i\theta} - e^{-i\theta})$$

The complex exponential function

$$e^{\alpha} = 1+\alpha + \frac{\alpha^2}{2!} + \ldots + \frac{\alpha^n}{n!} + \ldots$$

General case $\alpha = z = a + ib, \quad a, b$ real

$$e^{\imath z} = (1 - \frac{z^2}{2!} + \frac{z^4}{4!} + \cdots) + \imath(z - \frac{z^3}{3!} + \cdots)$$
$$= \cos z + \imath \sin z$$

Similarly one has

$$\cos z = \frac{1}{2}(e^{\imath z} + e^{-\imath z})$$

$$\sin z = \frac{1}{2i}(e^{\imath z} - e^{-\imath z})$$

Hence

$$\cos(\imath b) = \frac{1}{2}(e^{-b} + e^{b}) = \cosh b$$

$$\sin(\imath b) = \frac{1}{2i}(e^{-b} - e^{b}) = \imath \sinh b$$

For the case $z = a + ib$

$$\cos z = \frac{1}{2}(e^{iz} + e^{-iz})$$

$$\sin z = \frac{1}{2i}(e^{iz} - e^{-iz})$$

$$\begin{aligned}\cos z &= \cos(a + ib) \\ &= \frac{1}{2}(e^{(ia-b)} + e^{(-ia+b)}) \\ &= \frac{1}{2}(e^{-b}(\cos a + i\sin a) + e^{b}(\cos a - i\sin a))\end{aligned}$$

i.e.

$$\cos z = \cos a \cosh b - i \sin a \sinh b.$$

and analogously

$$\sin z = \sin a \cosh b + i \cos a \sinh b.$$

Complex exponential and trig identities

$$\cos(a+b)+\imath\sin(a+b)=\mathrm{e}^{\imath(a+b)}=\mathrm{e}^{\imath a}\mathrm{e}^{\imath b}$$
$$=(\cos a+\imath\sin a)(\cos b+\imath\sin b)$$
$$=(\cos a\cos b-\sin a\sin b)+\imath(\cos a\sin b+\sin a\cos b)$$

Equating real and imaginary parts

$$\cos(a+b)=\cos a\cos b-\sin a\sin b$$
$$\sin(a+b)=\sin a\cos b+\cos a\sin b$$

The complex logarithm

$\ln z$

$$e^{\ln z} = z \quad = |z| e^{\imath\theta} = e^{\ln|z|} e^{\imath\theta} = e^{\ln|z| + \imath\theta}$$

$$\Rightarrow \quad \boxed{\ln z = \ln|z| + \imath \arg(z)}$$

Need to know θ including 2πn phase ambiguity in z