ODEs and complex numbers

(+ Normal Modes, Wave Motion and the Wave Equation)

G.G.Ross, Oxford, Michaelmas 2008 (Hilary 2009)

Books:

Mathematical Methods for Physics and Engineering,
Riley, Hobson, Bence CUP

Mathematical Methods in the Physicsl Sciences.
Boas, Wiley

http://www.physics.ox.ac.uk/users/ross/

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This term : Lectures 1-4 Complex numbers, 5-9 ODEs

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Introduction

Why complex nos?

- Natural numbers (positive integers) 1, 2, 3, . . .
- Negative integers e.g. $20 + y = 12 \Rightarrow y = -8$
- Rationals e.g. $4x = 6 \Rightarrow x = \frac{3}{2}$
- Irrationals e.g. $x^2 = 2 \Rightarrow x = \sqrt{2}$
- Complex nos e.g. $x^2 = -1 \Rightarrow x = i \equiv \sqrt{-1}$

Complex numbers

$$z = a + ib$$

$(i^2 = -1)$

where a and b are real

$$a = \mathrm{Re}(z),\ b = \mathrm{Im}(z)$$

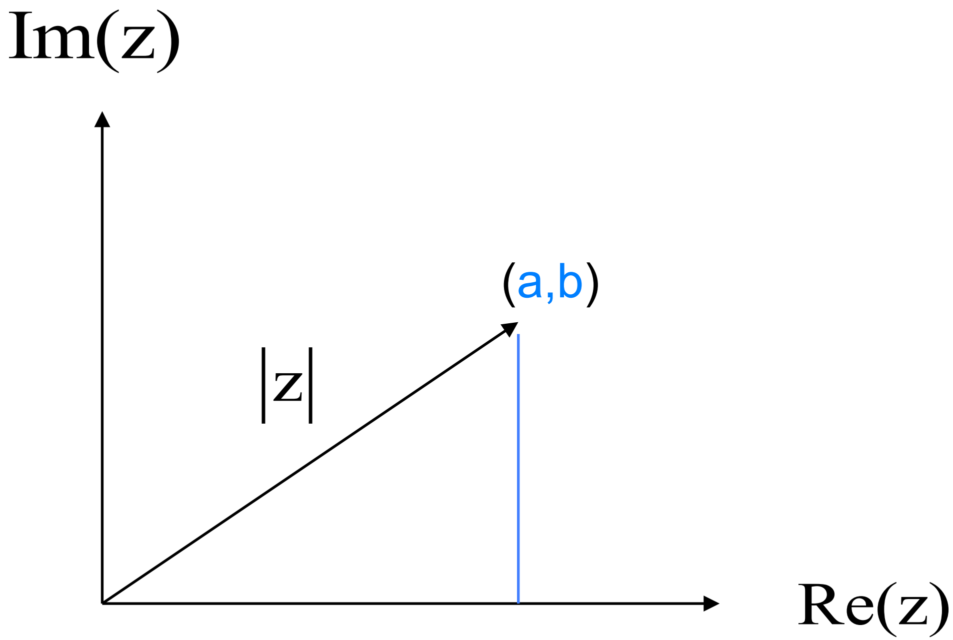
"Real part" "Imaginary part"

(Multiples of i (a=0) are called "pure imaginary" numbers.)

Argand diagram

Each z=a+ib $\rightarrow$ point (a, b) in plane:

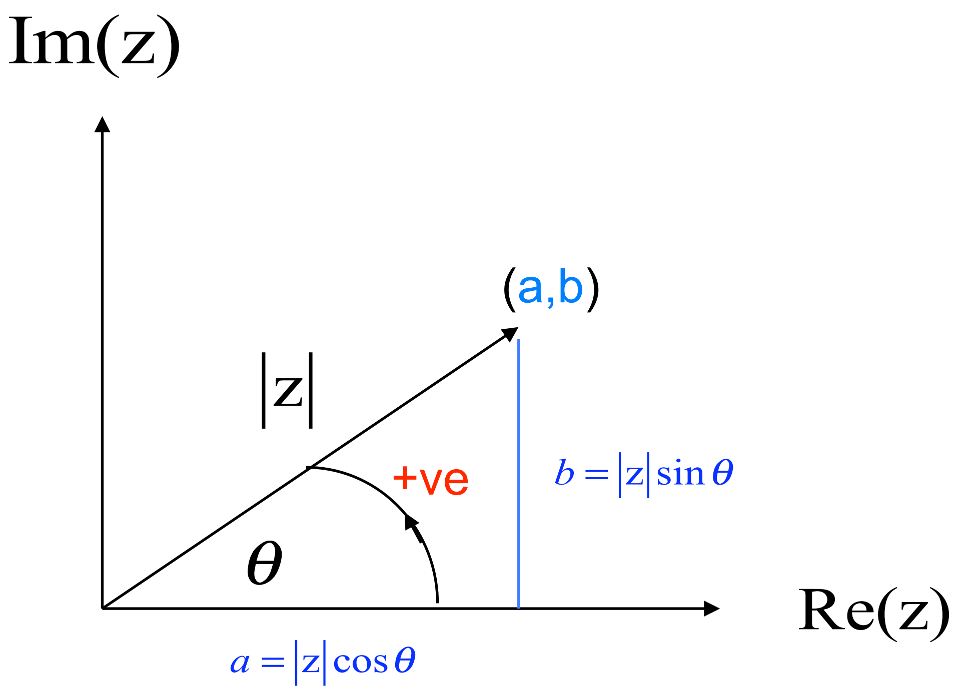
$$|z| \equiv \sqrt{a^2 + b^2}$$ length or "modulus"



Argand diagram

In polar co-ordinates: (r, θ)

$$|z| \equiv r = \sqrt{a^2 + b^2} \text{ "modulus"}$$



$$z = |z|(\cos\theta + i \sin\theta) \equiv r(\cos\theta + i \sin\theta)$$

Which quadrant?

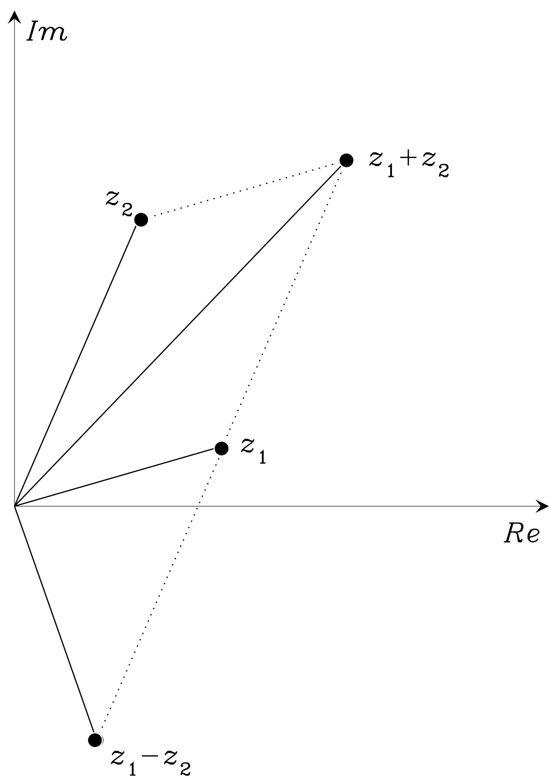
$$\theta \equiv \arg(z) = \arctan(b/a) \equiv \tan^{-1}(b/a)$$

"argument"

Addition :

$$z = a + ib$$

$$z_1 \pm z_2 = (a_1 \pm a_2) + i(b_1 \pm b_2)$$



Multiplication

$$z = a + ib$$

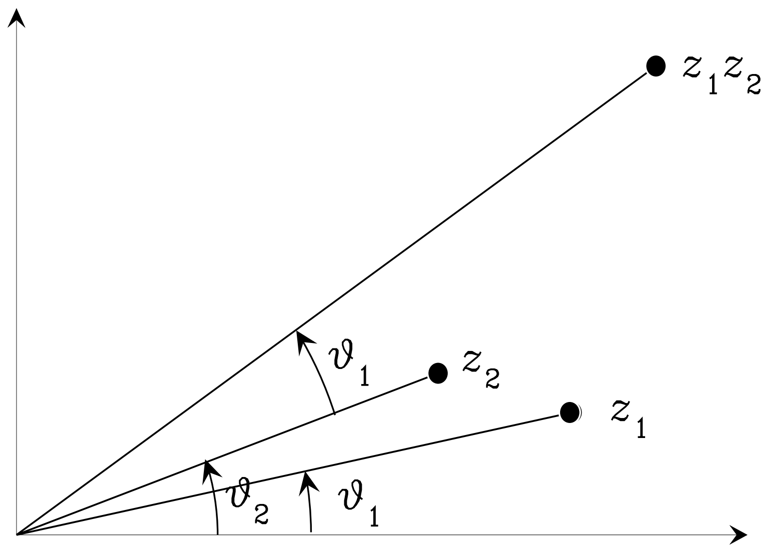
$$z_1 z_2 = (a_1 + ib_1)(a_2 + ib_2)$$
$$= (a_1 a_2 - b_1 b_2) + i(a_1 b_2 + b_1 a_2)$$

Multiplication

Polar co-ordinates

$$z_1 = r_1 \,(\cos\theta_1 + i\sin\theta_1)$$

$$z_2 = r_2 \,(\cos\theta_2 + i\sin\theta_2)$$



$$z_1 \, z_2 = r_1 \, r_2 \left(\cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2)\right)$$

$$| z_1 z_2 | = | z_2 | | z_2 |$$

$$Arg[z_1 z_2] = Arg[z_1] + Arg[z_2]$$

Division

$$\frac{z_1}{z_2}\ ?$$

$z = a + ib$

Define "complex conjugate"

$z^* = a - ib$

$$\text{Modulus}^2 \ : \ |z|^2 \equiv zz^* = (a^2 + b^2) \text{ is real (and } > 0)$$

$$\frac{1}{z_2} = \frac{1}{z_2}\frac{z_2^*}{z_2^*} = \frac{1}{|z_2|^2} z_2^*$$

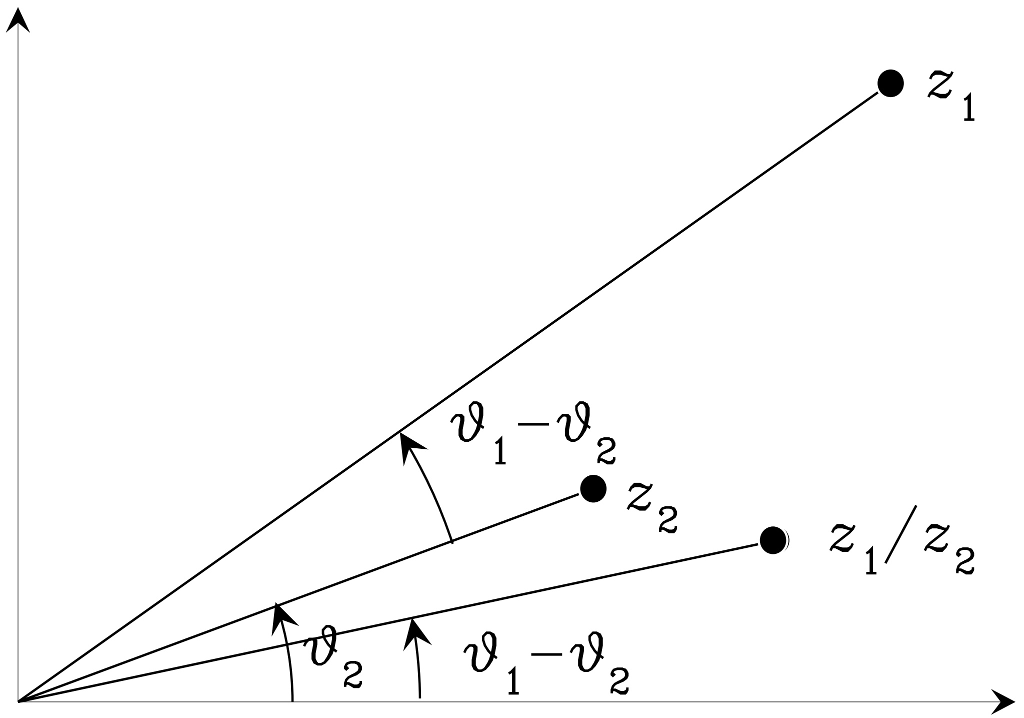
$$\frac{z_1}{z_2} = \frac{z_1 z_2^*}{|z_2|^2}$$

Division

Polar co-ordinates

$$z_1 = r_1 (\cos\theta_1 + i\sin\theta_1)$$

$$z_2 = r_2 (\cos\theta_2 + i\sin\theta_2)$$



$$\frac{z_1}{z_2} = \frac{r_1}{r_2} \left(\cos(\theta_1 - \theta_2) + i\sin(\theta_1 - \theta_2)\right)$$

e.g. Find the modulus $$\left|\frac{z_1}{z_2}\right| \equiv \frac{r_1}{r_2}$$

when $$\begin{cases} z_1 = 1+2i \\ z_2 = 1-3i \end{cases}$$

Elegant method:

$$\left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|} = \frac{\sqrt{1+4}}{\sqrt{1+9}} = \frac{1}{\sqrt{2}}$$

Clumsy method:

$$\left|\frac{z_1}{z_2}\right| = \left|\frac{1+2i}{1-3i}\right| = \frac{|z_1 z_2^*|}{|z_2|^2}$$

$$= \frac{|(1+2i)(1+3i)|}{1+9} = \frac{|(1-6)+i(2+3)|}{10}$$

$$= \frac{\sqrt{25+25}}{10} = \frac{\sqrt{2}}{2} = \frac{1}{\sqrt{2}}$$