FIRST YEAR MATHS FOR PHYSICS STUDENTS

NORMAL MODES AND WAVES

Hilary Term 2009

Prof. G. G. Ross

Question Sheet 3: Waves 2

[Questions marked with an asterisk (*) are optional]

1. (a) A string of uniform linear density ρ is stretched to a tension T, its ends being fixed at $x = 0$ and $x = L$. If $y(x,t)$ is the transverse displacement of the string at position x and time t, show that $c^2\partial^2 y/\partial x^2 = \partial^2 y/\partial t^2$ where $c^2 = T/\rho$. What is meant by the statement that this equation is '*linear*'?
Verify that

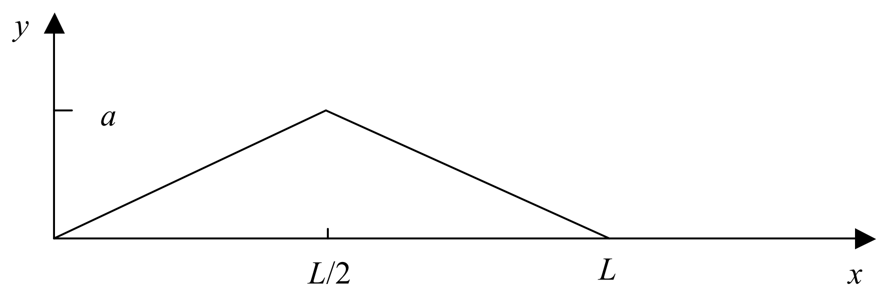
$$y(x,t) = A_r \sin\left(\frac{r\pi x}{L}\right)\sin\left(\frac{r\pi ct}{L}\right)$$

and

$$y(x,t) = B_r \sin\left(\frac{r\pi x}{L}\right)\cos\left(\frac{r\pi ct}{L}\right)$$

where r is any integer, are both solutions of this equation, obeying the boundary conditions $y(0,t) = y(L,t) = 0$. Explain why sums of such solutions are also solutions.

(b) The string is such that at $t = 0$, $\partial y/\partial t = 0$ for all x, and $y(x,0)$ has the shape



i.e. the mid-point is drawn aside a small distance a. Explain why the solution after the mid-point is released has the form

$$y(x,t) = \sum_{r=1}^{\infty} B_r \sin\left(\frac{r\pi x}{L}\right)\cos\left(\frac{r\pi ct}{L}\right)$$

[N.B. There are infinitely many constants (B_1, B_2, …) in this expression. They can be determined from the initial displacement of the string by the technique of Fourier Analysis:

$$y(x,0)=\sum_{r=1}^{\infty} B_r \sin\left(\frac{r\pi x}{L}\right) \qquad \text{hence } B_r=\frac{2}{L}\int_0^L y(x,0)\sin\left(\frac{r\pi x}{L}\right)dx\,.$$

2. Outline the solution of the wave equation

$$\frac{\partial^2 y}{\partial x^2}=\frac{1}{c^2}\frac{\partial^2 y}{\partial t^2}$$

using the method of *separation of variables*. Assuming that transverse waves exist on a string fixed between the points $x = 0$ and $x = l$. Assuming that at time t=0, the string is initially in its equilibrium position, but has a velocity

$$\frac{\partial y(x,t=0)}{\partial t}=V(x)$$

find, by the method of separation of variables, the general solution to the subsequent motion of the string.

3. (a) Stationary waves $y = f(x)g(t)$ exist on a string of length L. If the x-dependence of the displacement y is

$$f(x)=A\sin(kx).$$

Using the general solutions given in question 1a, find an expression for k. What is the t-dependence of $g(t)$? [This involves 2 arbitrary constants].

(b) At $t = 0$, the displacement is

$$y(x,0)=\sin\frac{\pi x}{L}+2\sin\frac{2\pi x}{L}$$

and the string is instantaneously stationary. Find the displacement at subsequent times.

Make rough sketches of $y(x,t)$ at the following times of t: 0, $L/4c$, $L/2c$, $3L/4c$, L/c.

4. Two semi-infinite strings are connected at $x = 0$ and stretched to a tension T. They have linear densities ρ_1 and ρ_2 respectively. A harmonic travelling wave, given in complex form as

$$A\exp\left[i\omega\left(t-x/v_1\right)\right]$$

travels along string 1 towards the boundary at $x = 0$. Determine the amplitudes of the reflected and transmitted waves.

Check that these amplitudes are such that energy conservation in the region at $x \approx 0$ is obeyed.

5. An infinite string of linear density ρ is under tension T, and has a mass m connected at $x = 0$. Calculate the amplitude reflection coefficient for transverse waves incident on the mass. What is the value of the amplitude reflection coefficient as m tends to zero?

6. A semi-infinite string of density ρ per unit length is under tension T. At its free end is a mass m which is constrained to move transversely. Determine the amplitude reflection coefficient for transverse waves incident on the mass. What is the phase difference between the incident and reflected waves?

7*. An infinite string lies along the x-axis, and is under tension T. It consists of a section at $0 < x < a$, of linear density ρ_1, and two semi-infinite pieces of density ρ_2. A moving wave $A\cos 2\pi\left(x/\lambda + vt\right)$ travels along the string at $x > a$, towards the short section.

How many types of waves are there in the various sections of the string? How many boundary conditions need to be satisfied?

Show that, if $a = n\lambda_1$, (where λ_1 is the wavelength on the short section, and n is an integer), the amplitude of the wave that emerges at $x < 0$ is A. What is the amplitude of the wave in the short section?

8*. For the infinite electrical circuit shown in the figure on page W35, show that the voltage V obeys the wave equation and determine the speed of the waves. Find the characteristic impedance Z (i.e. the ratio of voltage to current) for waves travelling in both the positive and negative x-direction. Why is the characteristic impedance positive for waves travelling to the right, but negative for waves travelling to the left? Isn't that paradoxical, since the circuit is the same for left- and right-travelling waves?

9*. A semi-infinite transmission line, of capacitance and inductance C and L per unit length, is terminated by an impedance Z_T (page W36). Find the ratio of the amplitude and the phase difference for the reflected and incident waves if (a) $Z_T = \sqrt{L/C}$, (b) $Z_T = 2\sqrt{L/C}$ or (c) Z_T is a capacitor of capacitance C. In (a) and (b) what type of impedance is required?