FIRST YEAR MATHS FOR PHYSICS STUDENTS

NORMAL MODES AND WAVES

Hilary Term 2009

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Question Sheet 2: Waves 1

1. At time $t = 0$, the displacement of an infinitely long string is defined as:

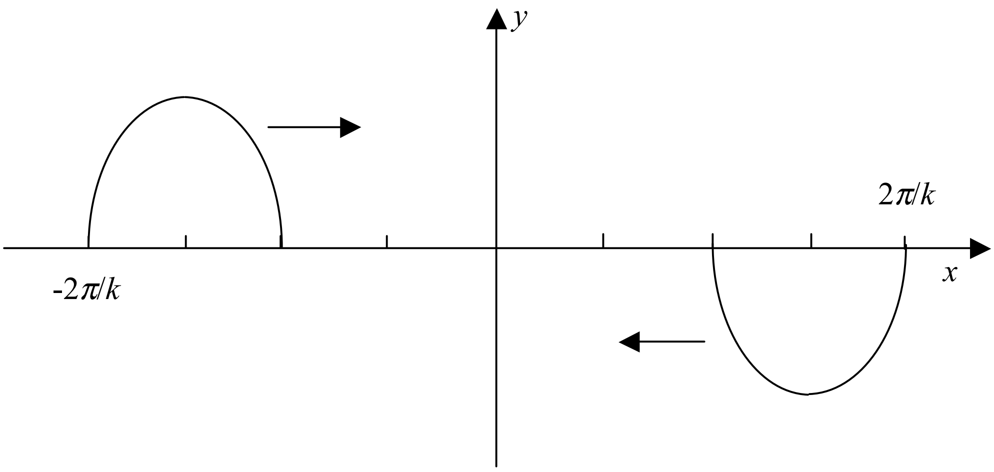
$$y(x, t=0) = \sin\frac{\pi x}{a} \text{ in the range } -a \le x \le a$$

and $$y(x, t=0) = 0 \text{ otherwise.}$$

The string is initially at rest.

Using d'Alembert's solution, and assuming that waves may move along the string with speed c, sketch the displacement of the string at $t = 0$, $t = a/2c$, and $t = a/c$.

2.



Two transverse waves are on the same piece of string. The first has displacement y non-zero only for $kx + \omega t$ between π and 2π, when it is equal to $A\sin(kx + \omega t)$. The second has $y = A\sin(kx - \omega t)$ for $kx - \omega t$ between -2π and $-\pi$, and is zero otherwise. When $t = 0$, the displacement is as shown in the figure. Calculate the energy of the two waves.

What is the displacement of the string at $t = 3\pi/2\omega$? Calculate the energy at this time.

3.(a) What is the difference between a moving wave and a stationary wave?

(b) Convince yourself that

$$y_1 = A\sin(kx - \omega t)$$

corresponds to a moving wave. Which way does it move? What are the amplitude, wavelength, frequency, period and velocity of the wave?

(c) Show that y_1 satisfies the wave equation

$$\frac{\partial^2 y}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 y}{\partial t^2}$$

provided that ω and k are suitably related.

(d) Write down a wave y_2 of equal amplitude travelling in the opposite direction. Show that $y_1 + y_2$ can be written in the form

$$y_1 + y_2 = f(x)g(t)$$

where $f(x)$ is a function of x only, and $g(t)$ is a function just of t. Convince yourself that the combination of two moving waves is a stationary wave. By determining $f(x)$ and $g(t)$ explicitly, determine the wavelength and frequency of $y_1 + y_2$. Comment on the velocity of the waves.

4. What is meant by (a) a dispersive medium, and (b) phase velocity, v? Explain the relevance of group velocity g for the transmission of signals in a dispersive medium. Justify the equation

$$g = \frac{d\omega}{dk} \qquad (1)$$

Show that alternative expressions for g are

$$g = \mathrm{v} + k\frac{dv}{dk} \qquad (2)$$

$$g = \mathrm{v} - \lambda\frac{dv}{d\lambda} \qquad (3)$$

$$g = \frac{c}{\mu}\left(1 + \frac{\lambda}{\mu}\frac{d\mu}{d\lambda}\right) \qquad (4)$$

where μ is the refractive index for waves of wavelength λ and wavenumber k (in the medium).

Use Eq. 3 to show that

$$g = \mathrm{v}\left[1 - 1\Big/\left(1 + \frac{\mathrm{v}}{\lambda'}\frac{d\lambda'}{d\mathrm{v}}\right)\right]$$

where λ' is the wavelength in vacuum.

5. In quantum mechanics, a particle of momentum p and energy E has associated with it a wave of wavelength λ and frequency f given by

$$\lambda = h/p \text{ and } f = E/h$$

where h is Planck's constant. Find the phase and group velocities of these waves given that

$$p = m_0\mathrm{v}/\sqrt{1 - \mathrm{v}^2/c^2} \text{ and } E = m_0c^2/\sqrt{1 - \mathrm{v}^2/c^2},$$

(The particle's rest mass is m_0, and its speed is v. c is the speed of light)
Comment on your answers.

6. Show that the kinetic energy U and the potential energy V for a length $\lambda = 2\pi / k$ of a transverse wave on a string of linear density ρ and at tension T are given by

$$U = \int_0^{\lambda} \frac{1}{2}\rho \left(\frac{\partial y}{\partial t}\right)^2 dx$$

and $$V = \int_0^{\lambda} \frac{1}{2}T \left(\frac{\partial y}{\partial x}\right)^2 dx$$

Evaluate these for the wave

$$y = A\cos\left(kx + \omega t + \phi\right)$$

and show that $U = V$.