CP4 revision:

ODE's and complex numbers
Normal Modes, Wave Motion and the Wave Equation

(Worked examples mainly from 2003, 2004 prelims papers)

Week 3: Tuesday, 10am
Week 4: Monday, 10 am, Thursday, 10am } Martin Wood

Optics (Prof J. Jones)

Week 4, Tuesday , 10am, Martin Wood

Functions of complex numbers

- The complex exponential

Functions defined by power series :

$$e^x = 1+x+\frac{x^2}{2!}+\ldots+\frac{x^n}{n!}+\ldots$$

$$\cos x = 1-\frac{x^2}{2!}+\frac{x^4}{4!}-\ldots+(-1)^n\frac{x^{2n}}{(2n)!}+\ldots$$

$$\sin x = x-\frac{x^3}{3!}+\frac{x^5}{5!}-\ldots+(-1)^n\frac{x^{2n+1}}{(2n+1)!}+\ldots$$

Define the complex exponential

$$e^\alpha = 1+\alpha+\frac{\alpha^2}{2!}+\ldots+\frac{\alpha^n}{n!}+\ldots$$

$\alpha = a+ib$

Special case $\alpha = i\theta$

$$e^{i\theta} = \left(1-\frac{\theta^2}{2!}+\frac{\theta^4}{4!}-\ldots\right)+i\left(\theta-\frac{\theta^3}{3!}+\frac{\theta^5}{5!}-\ldots\right)$$

$$= \cos\theta + i\sin\theta$$

The complex logarithm

$$\ln z$$

$$e^{\ln z} = z \quad = |z| e^{\imath\theta} = e^{\ln|z|} e^{\imath\theta} = e^{\ln|z| + \imath\theta}$$

$$\Rightarrow \quad \boxed{\ln z = \ln|z| + \imath \arg(z)}$$

Need to know θ including 2πn phase ambiguity in z

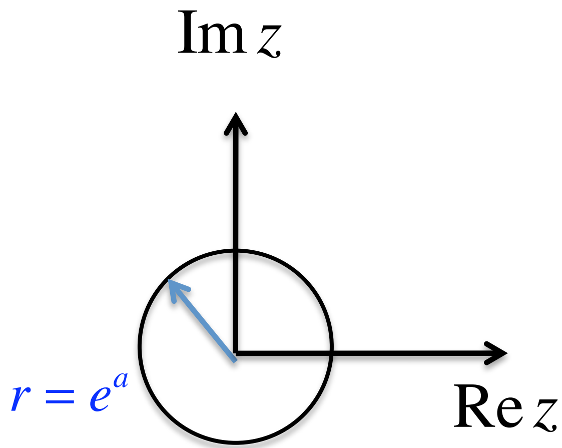
June 2003

1. Given that a is a real positive constant and z is a complex number, sketch the representations of the following on an Argand diagram. Label your axes clearly.

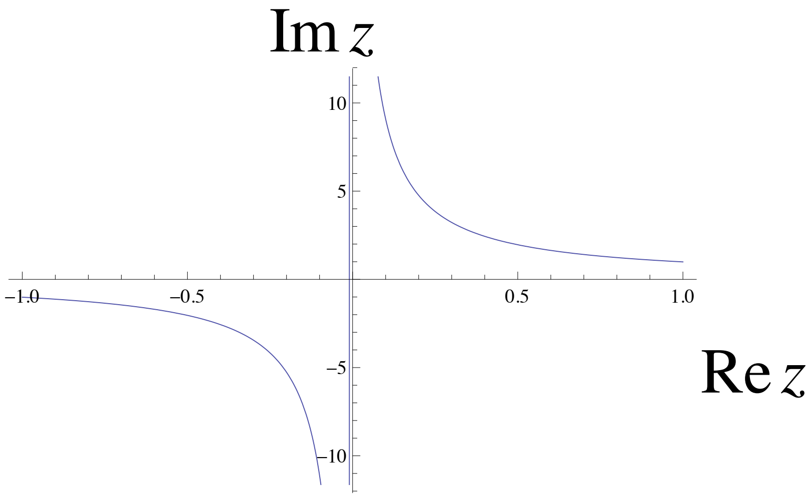
(a) $\mathrm{Re}\{\ln z\} = a$,

(b) $\mathrm{Im}\{z^2\} = 2a^2$.

$$(a) \quad z = re^{i\theta}, \quad \ln r = a, \quad r = e^a$$



$$(b) \quad z = x + iy, \quad 2xy = 2a^2, \quad y = \frac{a^2}{x}$$



June 2004

2. Using complex numbers or otherwise, evaluate the integral $\int_0^y e^{-bx} \sin ax \, dx$.

$$\int_0^y e^{-bx} \sin ax \, dx = \mathrm{Im}\{\int_0^y e^{-bx} e^{iax} \, dx = \mathrm{Im}\left[\frac{e^{(-b+ia)x}}{-b+ia}\right]_0^y = \mathrm{Im}\frac{e^{-by}(-b-ia)(\cos ax + i\sin ax)}{a^2+b^2}$$

$$= \frac{-e^{-by}}{a^2+b^2}(a\cos ax + b\sin ax)$$

September 2003

2. Solve the equation

$$z^6 + z^3 + 1 = 0.$$

$$u = z^3, \quad u^2 + u + 1 = 0, \quad u = \frac{-1 \pm \sqrt{3}i}{2} = e^{i(2\pi/3 + n\pi)}, \quad n = 0,1$$

$$z = u^{1/3} = e^{i(2\pi/9 + n\pi/3 + 2m\pi/3)}, \quad n = 0,1, \quad m = 0,1,2$$

$2b$. Solve the equation $z^6 - 15z^4 + 15z^2 - 1 = 0$

$(x+y)^0$	1
$(x+y)^1$	1 1
$(x+y)^2$	1 2 1
$(x+y)^3$	1 3 3 1
$(x+y)^4$	1 4 6 4 1
$(x+y)^5$	1 5 10 10 5 1

$$(z+i)^6 = z^6 + 6iz^5 - 15z^4 - 20iz^3 + 15z^2 + 6iz - 1$$

$$(z+i)^6 + (z-i)^6 = 2\left(z^6 - 15z^4 + 15z^2 - 1\right) = 0 \quad \Rightarrow \quad \frac{(z+i)^6}{(z-i)^6} \equiv u^6 = -1 = e^{i\pi}$$

$$u = e^{i(\pi/6 + 2\pi n/6)}, \quad n = 0,1,2,3,4,5$$

$$z - i = u(z+i) \quad \Rightarrow \quad z = \frac{i(1+u)}{(1-u)}$$

First order linear equations

General form : $\dfrac{\mathrm{d}f}{\mathrm{d}x} + q(x) f = h(x).$

Easy to solve

Integrating factor

Look for a function I(x) such that $\mathrm{I(x)}\dfrac{\mathrm{d}f}{\mathrm{d}x} + I(x)q(x) f \equiv \dfrac{dIf}{dx} = I(x)h(x)$

$$I(x) = e^{\int^{x} q(x')\,dx'}$$

Solution : $f(x) = \dfrac{1}{I(x)} \displaystyle\int_{x_0}^{x} I(x')h(x')dx'$

First order nonlinear equations

Although no general method for solution is available, there are several cases of physically relevant nonlinear equations which can be solved analytically :

Separable equations

$$\frac{dy}{dx} = \frac{f(x)}{g(y)}$$

Solution : $\int g(y)dy = \int f(x)dx$

Almost separable equations

$$\frac{dy}{dx} = f(ax+by)$$

Change variables : $z = ax + by$

$$\frac{dz}{dx} = a + bf(z)$$ Separable

Homogeneous equations

$$\frac{dy}{dx} = f(y/x).$$

Change variables : $y = \mathrm{v}x$

$$\frac{d\mathrm{v}}{dx} = \frac{1}{x}(f(\mathrm{v}) - \mathrm{v})$$ Separable

Homogeneous but for constants

$$\frac{dy}{dx}=\frac{x+2y+1}{x+y+2}$$

Change variables : $x = x'+a,\ y = y'+b$

$$\frac{dy'}{dx'}=\frac{x'+2y'+1+a+2b}{x'+y'+2+a+b}=\frac{x'+2y'}{x'+y'},$$ $a=-3,\ b=1$ Homogeneous

The Bernoulli equation

$$\frac{\mathrm{d}y}{\mathrm{d}x}+P(x)y=Q(x)y^{n}, \qquad n\neq 1$$

Change variables : $z = y^{1-n}$

$$\frac{dz}{dx}+(1-n)P(x)z=(1-n)Q(x),$$ First order linear

3. Solve the differential equation (September 2004)

$$\frac{\mathrm{d}y}{\mathrm{d}x} - \frac{y}{x} = x,$$

Ist order linear

given that $y = 2$ when $x = 1$.

1. Solve the differential equation (June 2005)

$$2\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{y(x+y)}{x^2}\,.$$

[6]

Homogeneous

8 (c) $y\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{x}{4x+3}.$ (June 2007)

Separable

Second order linear equation with constant coefficients

$$Lf = a_2 \frac{\mathrm{d}^2 f}{\mathrm{d}x^2} + a_1 \frac{\mathrm{d}f}{\mathrm{d}x} + a_0 f = h(x).$$

Complementary function

$$Lf_0 = a_2 \frac{\mathrm{d}^2 f_0}{\mathrm{d}x^2} + a_1 \frac{\mathrm{d}f_0}{\mathrm{d}x} + a_0 f_0 = 0.$$

Try $f_0 = \mathrm{e}^{mx}$ ⇒ $a_2 m^2 + a_1 m + a_0 = 0.$

"Auxiliary" equation

$$m_\pm \equiv \frac{-a_1 \pm \sqrt{a_1^2 - 4a_2 a_0}}{2a_2},$$

$a_1^2 - 4a_2 a_0 \to +, 0, -$

⇒ Complementary function $f_0 = A_+ \mathrm{e}^{m_+ x} + A_- \mathrm{e}^{m_- x}.$

Two constants of integration

Second order linear equation with constant coefficients

$$Lf = a_2 \frac{\mathrm{d}^2 f}{\mathrm{d}x^2} + a_1 \frac{\mathrm{d}f}{\mathrm{d}x} + a_0 f = h(x).$$

Complementary function

$$Lf_0 = a_2 \frac{\mathrm{d}^2 f_0}{\mathrm{d}x^2} + a_1 \frac{\mathrm{d}f_0}{\mathrm{d}x} + a_0 f_0 = 0.$$

Try $f_0 = \mathrm{e}^{mx}$ ⇒ $a_2 m^2 + a_1 m + a_0 = 0.$

$$m_{\pm} \equiv \frac{-a_1 \pm \sqrt{a_1^2 - 4a_2 a_0}}{2a_2},$$

"Auxiliary" equation

$a_1^2 - 4a_2 a_0 \to +, 0, -$

⇒ Complementary function

$$f_0 = A\mathrm{e}^{mx} + Bx\mathrm{e}^{mx}$$

Repeated roots

Second order linear equation with constant coefficients

$$Lf = a_2 \frac{\mathrm{d}^2 f}{\mathrm{d}x^2} + a_1 \frac{\mathrm{d}f}{\mathrm{d}x} + a_0 f = h(x).$$

Complementary function

$$Lf_0 = a_2 \frac{\mathrm{d}^2 f_0}{\mathrm{d}x^2} + a_1 \frac{\mathrm{d}f_0}{\mathrm{d}x} + a_0 f_0 = 0.$$

Particular integral

$$Lf_1 = a_2 \frac{\mathrm{d}^2 f_1}{\mathrm{d}x^2} + a_1 \frac{\mathrm{d}f_1}{\mathrm{d}x} + a_0 f_1 = h.$$

General solution : $f_0 + f_1$

11. A simple pendulum comprises a spherical bob of radius r and density ρ attached to one end of a light rigid rod of length ℓ which hangs vertically under gravity. The bob, which is immersed in a liquid of viscosity η and density ρ_0, is displaced from its equilibrium position at time $t = 0$ by a small angle α_0 to the vertical and released. The damping force is given by Stokes' formula, $F = 6\pi\eta r v$, where v is the velocity and η the viscosity. Show that the equation of motion of the bob is

$$\ddot{\alpha} + \left(\frac{9\eta}{2r^2\rho}\right)\dot{\alpha} + \left(\frac{\rho - \rho_0}{\rho}\right)\frac{g}{\ell}\alpha = 0.$$

[Archimedes' principle states that a body immersed in a fluid experiences an upthrust equal to the weight of the fluid displaced.]

Solve the equation to obtain an expression for the angular displacement $\alpha(t)$. [9]

Explain what conditions are necessary for the oscillation to be: (a) under damped, and (b) critically damped. Show that for critical damping

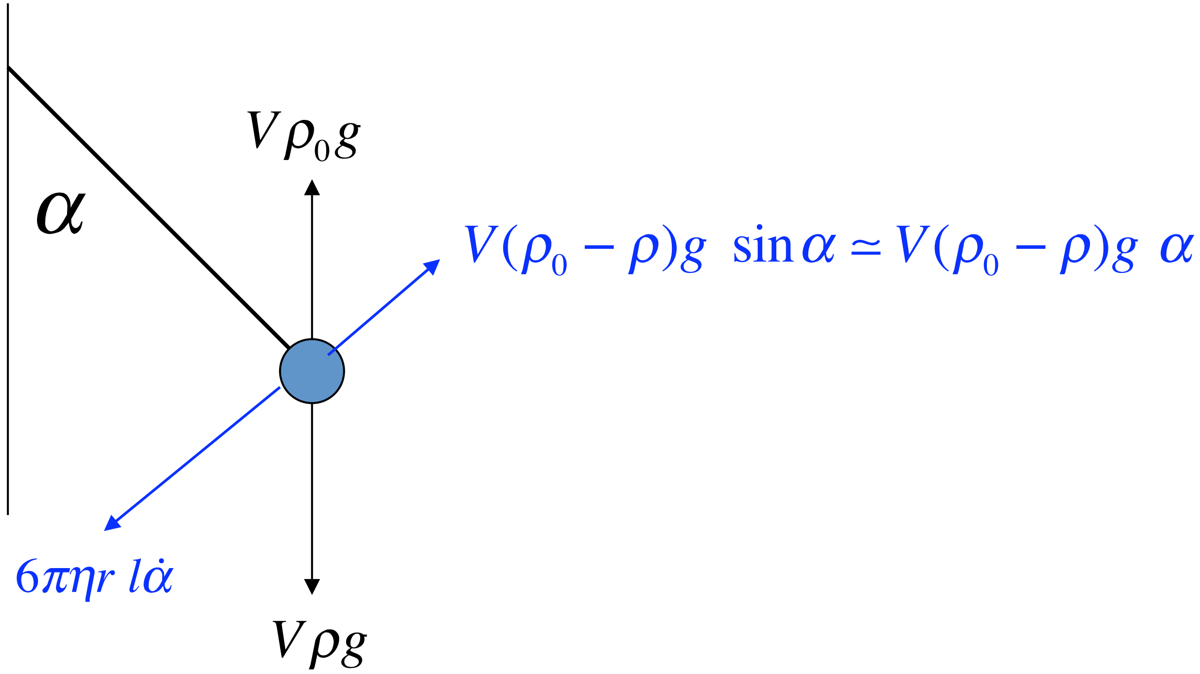
$$r_c = \frac{3}{2}\left[\frac{\eta^2\ell}{\rho\,(\rho - \rho_0)\,g}\right]^{\frac{1}{4}}.$$

For $r > r_c$ (but with all other quantities fixed) state whether the system is over or under damped. [7]

Calculate the radius the bob must have for the motion to be critically damped when $\ell = 0{\cdot}75\,\mathrm{m}$ and it is immersed in water. [4]

[Take the viscosity of water to be $\eta = 10^{-3}\,\mathrm{kg\,m^{-1}\,s^{-1}}$, the density of the bob to be $\rho = 5 \times 10^3\,\mathrm{kg\,m^{-3}}$ and the acceleration due to gravity $g = 9{\cdot}8\,\mathrm{ms^{-2}}$.]

June 2003 Q11 Phys



Newton's 2nd law in direction of motion:

$$V\rho\, l\ddot{\alpha} = -6\pi\eta r\ l\dot{\alpha} + V(\rho_0-\rho)g\alpha$$

$$V = \frac{4}{3}\pi r^3$$

$$\ddot{\alpha} + \frac{6\pi\eta r}{V\rho}\dot{\alpha} + \frac{(\rho_0-\rho)}{\rho}\frac{g}{l}\alpha = 0$$

$$\boxed{\ddot{\alpha} + \frac{9\eta}{2r^2\rho}\dot{\alpha} + \frac{(\rho_0-\rho)}{\rho}\frac{g}{l}\alpha = 0}$$

June 2003 Q11 Phys

$$\ddot{\alpha} + A\,\dot{\alpha} + B\,\alpha = 0$$

$$\ddot{\alpha} + \frac{9\eta}{2r^2\rho}\dot{\alpha} + \frac{(\rho_0 - \rho)}{\rho}\frac{g}{l}\alpha = 0$$

$C.F.$ Try α=a e^{mt}

Auxiliary equation : $m^2 + Am + Bm = 0$

$$m_{1,2} = \frac{-A \pm \left(A^2 - 4B\right)^{\frac{1}{2}}}{2}$$

$$\alpha(t) = Ae^{m_1 t} + Be^{m_2 t}$$

a). Underdamped: $A^2 < 4B$

b). Critically damped $A^2 = 4B$

$$\left(\frac{9\eta}{2r_c^{\,2}\rho}\right)^2 = 4\frac{g}{l}\frac{(\rho_0 - \rho)}{\rho} \quad \Rightarrow \quad r_c = \frac{3}{2}\left(\frac{\eta^2 l}{\rho(\rho - \rho_0)g}\right)^{1/4}$$

$r > r_c$ underdamped