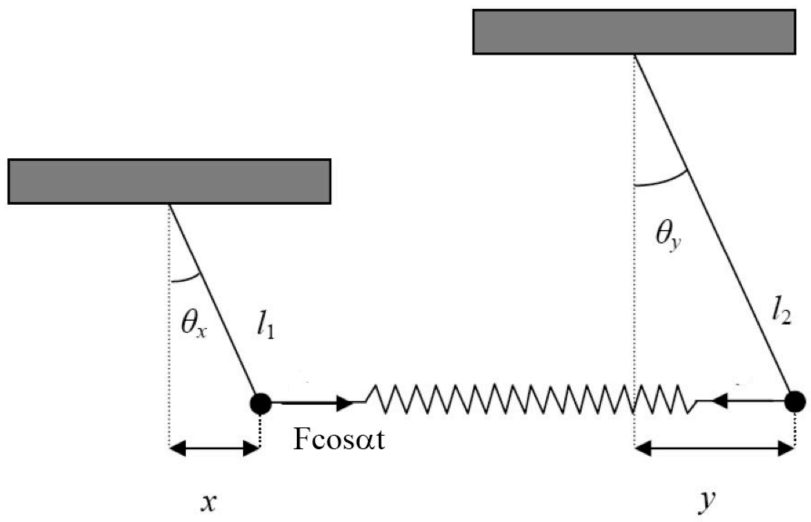
The damped driven pendulum - the Particular Integral



$$m_1\ddot{x} = -\gamma\dot{x} - m_1 g x / l_1 + k\left(y - x\right) + F\cos\alpha t$$

$$m_2\ddot{y} = -\gamma\dot{y} - m_2 g y / l_2 - k\left(y - x\right)$$

$$\begin{pmatrix} \frac{d^2}{dt^2} + \frac{\gamma}{m_1}\frac{d}{dt} + \left(\frac{g}{l_1} + \frac{k}{m_1}\right) & -\frac{k}{m_1} \\ -\frac{k}{m_2} & \frac{d^2}{dt^2} + \frac{\gamma}{m_2}\frac{d}{dt} + \left(\frac{g}{l_2} + \frac{k}{m_2}\right) \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \frac{F}{m_1}\begin{pmatrix} 1 \\ 0 \end{pmatrix}\mathrm{Re}\left(e^{i\alpha t}\right)$$

$$\begin{pmatrix} \frac{d^2}{dt^2}+\frac{\gamma}{m_1}\frac{d}{dt}+\left(\frac{g}{l_1}+\frac{k}{m_1}\right) & -\frac{k}{m_1} \\ -\frac{k}{m_2} & \frac{d^2}{dt^2}+\frac{\gamma}{m_2}\frac{d}{dt}+\left(\frac{g}{l_2}+\frac{k}{m_2}\right) \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}=\frac{F}{m_1}\begin{pmatrix} 1 \\ 0 \end{pmatrix}\mathrm{Re}\left(e^{i\alpha t}\right)$$

PI

$$\begin{pmatrix} x \\ y \end{pmatrix}=\mathrm{Re}\left(\begin{pmatrix} P \\ Q \end{pmatrix}e^{i\alpha t}\right)$$

$$\begin{pmatrix} -\alpha^2+i\frac{\gamma}{m_1}\alpha+\left(\frac{g}{l_1}+\frac{k}{m_1}\right) & -\frac{k}{m_1} \\ -\frac{k}{m_2} & -\alpha^2+i\frac{\gamma}{m_2}\alpha+\left(\frac{g}{l_2}+\frac{k}{m_2}\right) \end{pmatrix}\begin{pmatrix} P \\ Q \end{pmatrix}\equiv \mathbf{MP}=\frac{F}{m_1}\begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$\mathbf{P}\equiv\begin{pmatrix} P \\ Q \end{pmatrix}=\mathbf{M}^{-1}\frac{F}{m_1}\begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$\mathbf{M}=\begin{pmatrix} -\alpha^2+i\frac{\gamma}{m_1}\alpha+\left(\frac{g}{l_1}+\frac{k}{m_1}\right) & -\frac{k}{m_1} \\ -\frac{k}{m_2} & -\alpha^2+i\frac{\gamma}{m_2}\alpha+\left(\frac{g}{l_2}+\frac{k}{m_2}\right) \end{pmatrix}$$

$$\begin{pmatrix} x \\ y \end{pmatrix}=\mathrm{Re}\left(\mathbf{M}^{-1}\frac{F}{m_1}\begin{pmatrix} 1 \\ 0 \end{pmatrix}e^{i\alpha t}\right)$$

$$\begin{pmatrix} x \\ y \end{pmatrix} = \mathrm{Re}\left(\mathbf{M}^{-1} \frac{F}{m_1} \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{i\alpha t} \right)$$

$$\mathbf{M} = \begin{pmatrix} -\alpha^2 + i\frac{\gamma}{m_1}\alpha + \left(\frac{g}{l_1} + \frac{k}{m_1}\right) & -\frac{k}{m_1} \\ -\frac{k}{m_2} & -\alpha^2 + i\frac{\gamma}{m_2}\alpha + \left(\frac{g}{l_2} + \frac{k}{m_2}\right) \end{pmatrix}$$

$$e.g. \quad m_1 = m_2 = m \quad l_1 = l_2 = l \quad \gamma = 0$$

$$\left(\omega_1^2 = \frac{g}{l}, \quad \omega_2^2 = \frac{g}{l} + 2\frac{k}{m} \right)$$

$$\mathbf{M} = \begin{pmatrix} -\alpha^2 + \left(\frac{g}{l} + \frac{k}{m}\right) & -\frac{k}{m} \\ -\frac{k}{m} & -\alpha^2 + \left(\frac{g}{l} + \frac{k}{m}\right) \end{pmatrix}$$

$$\mathbf{M}^{-1} = \frac{1}{DetM} \begin{pmatrix} -\alpha^2 + \left(\frac{g}{l} + \frac{k}{m}\right) & \frac{k}{m} \\ \frac{k}{m} & -\alpha^2 + \left(\frac{g}{l} + \frac{k}{m}\right) \end{pmatrix}$$

$$DetM = \left(-\alpha^2 + \left(\frac{g}{l} + \frac{k}{m}\right)\right)^2 - \left(\frac{k}{m}\right)^2 = \left(\alpha^2 - \omega_1^2\right)\left(\alpha^2 - \omega_2^2\right)$$

$$\begin{pmatrix} x \\ y \end{pmatrix} = \left(\frac{F}{m}\cos\alpha t\right) \frac{1}{\left(\alpha^2 - \omega_1^2\right)\left(\alpha^2 - \omega_2^2\right)} \begin{pmatrix} -\alpha^2 + \left(\frac{g}{l} + \frac{k}{m}\right) \\ \frac{k}{m} \end{pmatrix} = \left(\frac{F}{m}\cos\alpha t\right) \frac{1}{\left(\alpha^2 - \omega_1^2\right)\left(\alpha^2 - \omega_2^2\right)} \begin{pmatrix} -\alpha^2 + \left(\omega_2^2 + \omega_1^2\right)/2 \\ \left(\omega_2^2 - \omega_1^2\right)/2 \end{pmatrix}$$

$$= \left(-\frac{F}{2m}\cos\alpha t\right) \begin{pmatrix} \frac{1}{\left(\alpha^2 - \omega_1^2\right)} + \frac{1}{\left(\alpha^2 - \omega_2^2\right)} \\ \frac{1}{\left(\alpha^2 - \omega_1^2\right)} - \frac{1}{\left(\alpha^2 - \omega_2^2\right)} \end{pmatrix}$$

Forced oscillations - decoupling method

$$m_1 = m_2 = m \quad l_1 = l_2 = l$$

$$m\ddot{x} + \gamma\dot{x} + m\frac{g}{l}x + k\left(x - y\right) = F\cos\alpha t$$
$$m\ddot{y} + \gamma\dot{y} + m\frac{g}{l}y + k\left(y - x\right) = 0$$

$$\left(\ddot{x} + \ddot{y}\right) + \frac{\gamma}{m}\left(\dot{x} + \dot{y}\right) + \frac{g}{l}\left(x + y\right) = \frac{F}{m}\cos\alpha t$$

$$\left(\ddot{x} - \ddot{y}\right) + \frac{\gamma}{m}\left(\dot{x} - \dot{y}\right) + \frac{g}{l}\left(x - y\right) + \frac{2k}{m}\left(x - y\right) = \frac{F}{m}\cos\alpha t$$

$$\ddot{q}_1 + \frac{\gamma}{m}\dot{q}_1 + \frac{g}{l}q_1 = \frac{F}{m}\cos\alpha t$$

$$\ddot{q}_2 + \frac{\gamma}{m}\dot{q}_2 + \left(\frac{g}{l} + \frac{2k}{m}\right)q_2 = \frac{F}{m}\cos\alpha t$$

$$q_1 = x + y$$
$$q_2 = x - y$$

CF already done, PI …

- $$\ddot{q}_1 + \frac{\gamma}{m}\dot{q}_1 + \frac{g}{l}q_1 = \frac{F}{m}\cos\alpha t = \mathrm{Re}\left[\frac{F}{m}\exp(i\alpha t)\right], \qquad q_1 = x + y$$

PI $$q_1 = \mathrm{Re}\left[A_1 \exp(i\alpha t)\right]$$

$$\left(-\alpha^2 + i\alpha\frac{\gamma}{m} + \frac{g}{l}\right)A_1 = F/m$$

$$A_1 = \frac{(F/m)\exp(i\phi_1)}{\left(\left(\frac{g}{l} - \alpha^2\right)^2 + \left(\frac{\alpha\gamma}{m}\right)^2\right)^{1/2}} = \frac{(F/m)\exp(i\phi_1)}{\left(\left(\omega_1^2 - \alpha^2\right)^2 + \left(\frac{\alpha\gamma}{m}\right)^2\right)^{1/2}}$$

where $\tan\phi_1 = \frac{-\alpha(\gamma/m)}{\left(\omega_1^2 - \alpha^2\right)}$ and $\omega_1 = \sqrt{\frac{g}{l}}$ the undamped normal mode frequency

$$q_1 = \frac{(F/m)}{\left(\left(\omega_1^2 - \alpha^2\right)^2 + \left(\frac{\alpha\gamma}{m}\right)^2\right)^{1/2}}\cos(\alpha t + \phi_1)$$

- $$\ddot{q}_2 + \frac{\gamma}{m}\dot{q}_2 + \left(\frac{g}{l} + \frac{2k}{m}\right)q_2 = \frac{F}{m}\cos\alpha t \qquad q_2 = x - y$$

PI $$q_2 = \mathrm{Re}\left[A_2 \exp(i\alpha t)\right]$$

$$A_2 = \frac{(F/m)\exp(i\phi_2)}{\left(\left(\frac{g}{l} + \frac{2k}{m} - \alpha^2\right)^2 + \left(\frac{\alpha\gamma}{m}\right)^2\right)^{1/2}} = \frac{(F/m)\exp(i\phi_2)}{\left((\omega_2^2 - \alpha^2)^2 + \left(\frac{\alpha\gamma}{m}\right)^2\right)^{1/2}}, \qquad \tan\phi_2 = \frac{-\alpha(\gamma/m)}{(\omega_2^2 - \alpha^2)}$$

$$q_2 = \frac{(F/m)}{\left((\omega_2^2 - \alpha^2)^2 + \left(\frac{\alpha\gamma}{m}\right)^2\right)^{1/2}}\cos(\alpha t + \phi_2)$$

$$q_1 = \frac{(F/m)}{\left((\omega_1^2 - \alpha^2)^2 + \left(\frac{\alpha\gamma}{m}\right)^2\right)^{1/2}}\cos(\alpha t + \phi_1)$$

Finally $x = \frac{1}{2}(q_1 + q_2)$, $y = \frac{1}{2}(q_1 - q_2)$

$$c.f. \begin{pmatrix} x \\ y \end{pmatrix} = \left(-\frac{F}{2m}\cos\alpha t\right)\begin{pmatrix} \frac{1}{(\alpha^2-\omega_1^2)} + \frac{1}{(\alpha^2-\omega_2^2)} \\ \frac{1}{(\alpha^2-\omega_1^2)} - \frac{1}{(\alpha^2-\omega_2^2)} \end{pmatrix}, \quad \gamma = 0$$

$$\begin{pmatrix} \frac{d^2}{dt^2}+\frac{\gamma}{m_1}\frac{d}{dt}+\left(\frac{g}{l_1}+\frac{k}{m_1}\right) & -\frac{k}{m_1} \\ -\frac{k}{m_2} & \frac{d^2}{dt^2}+\frac{\gamma}{m_2}\frac{d}{dt}+\left(\frac{g}{l_2}+\frac{k}{m_2}\right) \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}=\frac{F}{m_1}\begin{pmatrix} 1 \\ 0 \end{pmatrix}\mathrm{Re}\left(e^{i\omega t}\right)$$

The case $m_1 = m_2 = m, \quad l_1 \neq l_2$, no damping, no driving force.

Eigenvalue equation

$$\begin{vmatrix} -\omega^2+\left(\frac{g}{l_1}+\frac{k}{m}\right) & -\frac{k}{m} \\ -\frac{k}{m} & -\omega^2+\left(\frac{g}{l_2}+\frac{k}{m}\right) \end{vmatrix}=0$$

$A = g/l_1 + k/m = \beta_1^2 + k/m$
$B = -k/m$
$C = g/l_2 + k/m = \beta_2^2 + k/m$

$$\begin{pmatrix} x \\ y \end{pmatrix}=\mathrm{Re}\left\{\begin{pmatrix} x_0 \\ y_0 \end{pmatrix}e^{i\omega t}\right\}$$

$$\omega_{1,2}^2=\frac{1}{2}\left[\left(\beta_1^2+\beta_2^2\right)+2k/m\pm\sqrt{\left(\beta_1^2-\beta_2^2\right)^2+\left(2k/m\right)^2}\right]$$

$$\frac{x_0}{y_0}=-\frac{m}{2k}\left[\left(\beta_1^2-\beta_2^2\right)\pm\sqrt{\left(\beta_1^2-\beta_2^2\right)^2+\left(2k/m\right)^2}\right]$$

$$\left(\frac{y_0}{x_0}\right)_1=-1/\left(\frac{y_0}{x_0}\right)_2\equiv r$$

$$\left(\frac{y_0}{x_0}\right)_1 = -1 / \left(\frac{y_0}{x_0}\right)_2 \equiv r$$

$$(x_0)_1 = De^{i\delta_1}, \quad (x_0)_2 = De^{i\delta_2}$$

$$\mathbf{x}(t) = \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ r \end{pmatrix} D\cos(\omega_1 t + \delta_1) + \begin{pmatrix} -r \\ 1 \end{pmatrix} G\cos(\omega_2 t + \delta_2)$$

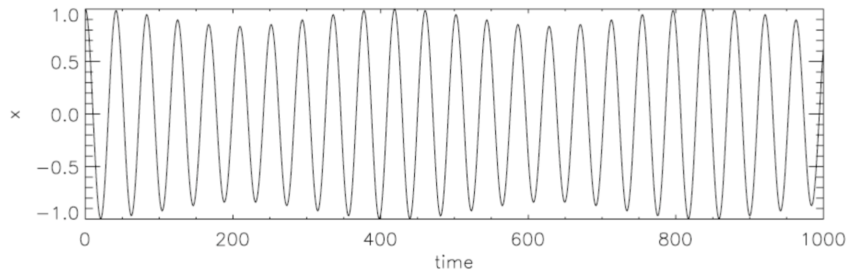
Initial conditions : $\mathbf{x}(0) = \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} a \\ 0 \end{pmatrix}, \quad \dot{\mathbf{x}} = 0$

$$\bar{\omega} = \frac{\omega_1 + \omega_2}{2}, \quad \Delta\omega = \omega_1 - \omega_2$$

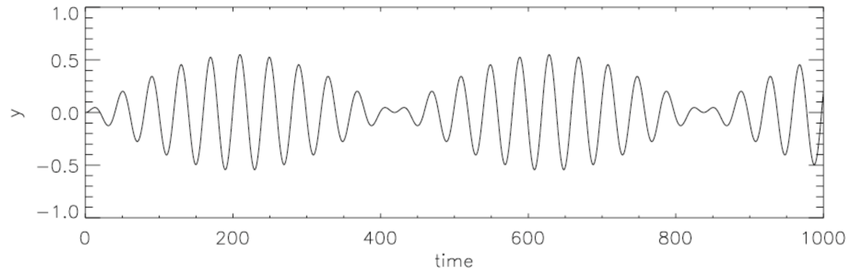
$$x(t) = a\left[\cos\omega_1 t + r^2 \cos\omega_2 t\right] / \left(1 + r^2\right)$$
$$y(t) = ar\left[\cos\omega_1 t - \cos\omega_2 t\right] / \left(1 + r^2\right)$$

$$\Rightarrow$$

$$x(t) = a\cos(\bar{\omega}t)\cos(\Delta\omega t / 2) - a\left(\frac{1 - r^2}{1 + r^2}\right)\sin(\bar{\omega}t)\sin(\Delta\omega t / 2)$$
$$y(t) = 2ar\sin(\bar{\omega}t)\sin(\Delta\omega t / 2) / \left(1 + r^2\right)$$



$$\left(\frac{1 - r^2}{1 + r^2}\right) a \le |x| \le a$$



$$0 \le y \le \left(\frac{2r}{1 + r^2}\right) a$$

Diagrammatic Representation of Normal Modes

$$\mathbf{v}_{1,2} = \left(x_0\mathbf{i} + y_0\mathbf{j}\right) / \sqrt{x_0^2 + y_0^2}$$

$$\left(\frac{x_0}{y_0}\right)_{1,2} = -\frac{m}{2k}\left[\left(\beta_1^2 - \beta_2^2\right) \pm \sqrt{\left(\beta_1^2 - \beta_2^2\right)^2 + \left(2k/m\right)^2}\,\right]$$

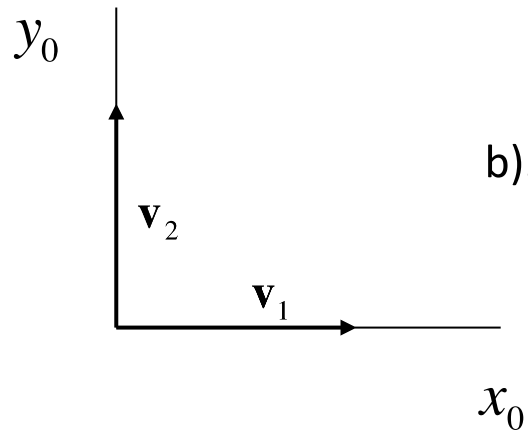
(a) For $k/m \to 0$ $x_0/y_0 \to$ $-\infty$ or 0

(b) For $k/m \to \infty$ $x_0/y_0 \to$ -1 or 1

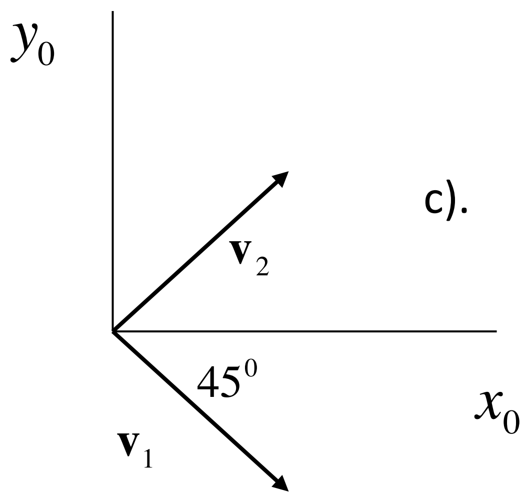
(c) Intermediate k/m

$$\frac{y_0}{x_0} = \tan\theta = \frac{-2k/m}{\left(\beta_1^2 - \beta_2^2\right) \pm \sqrt{\left(\beta_1^2 - \beta_2^2\right)^2 + \left(2k/m\right)^2}}$$

a).



b).



c).