NORMAL MODES, WAVE MOTION AND THE WAVE EQUATION

G.G.Ross, Hilary term 2009

12 lectures : 5 Normal Modes
7 Waves

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Lecture notes, overheads and problem sheets may be found at
http://www.physics.ox.ac.uk/users/ross/

Textbooks
Normal Modes : Riley, Hobson, Spence; Boas Waves", by

"Waves", by C.A.Coulson & A. Jeffrey, Longman
"Vibrations & Waves" by A.P.French, MIT Introductory Physics Series

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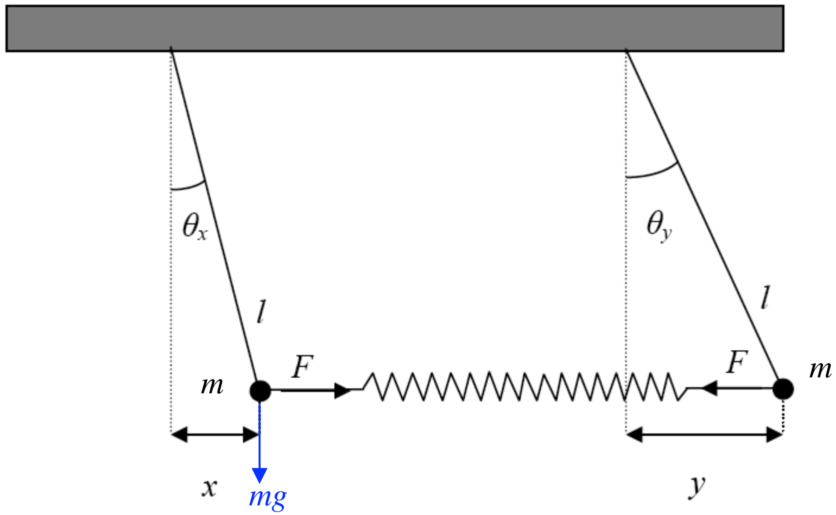
…see updated overhead : Power in oscillator

$$P = \frac{\partial W}{\partial t} = F\dot{x} \qquad W = \int_{x(t_0)}^{x(t)} F\,dx'$$

Normal Modes

Coupled differential equations - e.g. coupled pendula



$$m\ddot{x} = -mg\frac{x}{l} + k(y - x)$$

$$m\ddot{y} = -mg\frac{y}{l} - k(y - x)$$

$$\ddot{x} + g\frac{x}{l} - \frac{k}{m}x + \frac{k}{m}y = 0$$
$$\ddot{y} + g\frac{y}{l} - \frac{k}{m}y + \frac{k}{m}x = 0$$

Coupled first order linear differential equations

Solution I - Matrix method :

$$\begin{pmatrix} \frac{d^2}{dt^2} + \frac{g}{l} + \frac{k}{m} & -\frac{k}{m} \\ -\frac{k}{m} & \frac{d^2}{dt^2} + \frac{g}{l} + \frac{k}{m} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

CF : Try

$$\begin{pmatrix} x \\ y \end{pmatrix} = \mathrm{Re} \begin{pmatrix} X \\ Y \end{pmatrix} e^{i\omega t}$$

X, Y (complex) constants

$$\begin{pmatrix} -\omega^2 + \frac{g}{l} + \frac{k}{m} & -\frac{k}{m} \\ -\frac{k}{m} & -\omega^2 + \frac{g}{l} + \frac{k}{m} \end{pmatrix} \begin{pmatrix} X \\ Y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} -\omega^2 + \frac{g}{l} + \frac{k}{m} & -\frac{k}{m} \\ -\frac{k}{m} & -\omega^2 + \frac{g}{l} + \frac{k}{m} \end{pmatrix} \begin{pmatrix} X \\ Y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$A.\Psi = 0 \qquad \Rightarrow \qquad Det[A] = 0$$

Eigenvalue equation

$$\begin{vmatrix} -\omega^2 + \frac{g}{l} + \frac{k}{m} & -\frac{k}{m} \\ -\frac{k}{m} & -\omega^2 + \frac{g}{l} + \frac{k}{m} \end{vmatrix} = 0$$

$$\left(-\omega^2 + \frac{g}{l} + \frac{k}{m}\right)^2 - \left(\frac{k}{m}\right)^2 = 0 \qquad \Rightarrow \qquad \left(-\omega^2 + \frac{g}{l} + \frac{k}{m}\right) = \pm\left(\frac{k}{m}\right)$$

$$\left(-\omega^2 + \frac{g}{l} + \frac{k}{m}\right) = \pm\left(\frac{k}{m}\right)$$

Eigenvalue equation

$$\omega_1^2 = \frac{g}{l} \quad \text{or} \quad \omega_2^2 = \frac{g}{l} + 2\frac{k}{m}$$

Eigenvalues

$$\begin{pmatrix} x \\ y \end{pmatrix} = \mathrm{Re}\begin{pmatrix} X \\ Y \end{pmatrix} e^{i\omega t} \Longrightarrow \begin{pmatrix} -\omega^2 + \frac{g}{l} + \frac{k}{m} & -\frac{k}{m} \\ -\frac{k}{m} & -\omega^2 + \frac{g}{l} + \frac{k}{m} \end{pmatrix} \begin{pmatrix} X \\ Y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Eigenvector equation

$$\underline{\omega = \omega_1}: \begin{pmatrix} +\frac{k}{m} & -\frac{k}{m} \\ -\frac{k}{m} & +\frac{k}{m} \end{pmatrix} \begin{pmatrix} X_1 \\ Y_1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Longrightarrow X_1 = Y_1 = A_1 e^{i\phi_1}$$

Eigenvectors

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} A_1 \cos(\omega_1 t + \phi_1)$$

1st Normal mode

$$\left(-\omega^2+\frac{g}{l}+\frac{k}{m}\right)=\pm\left(\frac{k}{m}\right)$$

Eigenvalue equation

$$\omega_1^2=\frac{g}{l} \quad \text{or} \quad \omega_2^2=\frac{g}{l}+2\frac{k}{m}$$

Eigenvalues

$$\begin{pmatrix} x \\ y \end{pmatrix}=\mathrm{Re}\begin{pmatrix} X \\ Y \end{pmatrix}e^{i\omega t} \Longrightarrow \begin{pmatrix} -\omega^2+\frac{g}{l}+\frac{k}{m} & -\frac{k}{m} \\ -\frac{k}{m} & -\omega^2+\frac{g}{l}+\frac{k}{m} \end{pmatrix}\begin{pmatrix} X \\ Y \end{pmatrix}=\begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Eigenvector equation

$$\omega=\omega_2 \quad \begin{pmatrix} -\frac{k}{m} & -\frac{k}{m} \\ -\frac{k}{m} & -\frac{k}{m} \end{pmatrix}\begin{pmatrix} X_2 \\ Y_2 \end{pmatrix}=\begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow X_2=-Y_2=A_2e^{i\phi_2}$$

Eigenvectors

$$\begin{pmatrix} x \\ y \end{pmatrix}=\begin{pmatrix} 1 \\ -1 \end{pmatrix}A_2\cos(\omega_2 t+\phi_2)$$

2nd Normal mode

$$m\ddot{x} = -mg\frac{x}{l} + k(y-x)$$

$$m\ddot{y} = -mg\frac{y}{l} - k(y-x)$$

General solution given by a superposition of the two independent (normal mode)solutions:

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} A_1 \cos(\omega_1 t + \phi_1) + \begin{pmatrix} 1 \\ -1 \end{pmatrix} A_2 \cos(\omega_2 t + \phi_2)$$

Normal Modes

$$x + y = 2A_1 \cos(\omega_1 t + \phi_1)$$

$$x - y = 2A_2 \cos(\omega_2 t + \phi_2)$$

} N 1D linear differential Equations…N normal modes