NORMAL MODES, WAVE MOTION AND THE WAVE EQUATION

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This course of twelve lectures covers material for the paper CP4: Differential Equations, Waves and Optics of the Preliminary Examination in Physics, and Moderations in Physics and Philosophy. Textbooks covering aspects of this course include "Waves", by C.A.Coulson & A. Jeffrey, Longman and "Vibrations & Waves" by A.P.French, MIT Introductory Physics Series. I am grateful to Dr. P G Irwin for providing his notes for this course; they have been very useful in preparing these lecture notes.

Part 1 NORMAL MODES

1 Introduction

Many physical systems require more than one variable to quantify their configuration; for example a circuit may have two connected current loops, so one needs to know what current is flowing in each loop at each moment. Another example is a set of N coupled pendula each of which is a one-dimensional (1-D) oscillator. A set of differential equations– one for each variable – will determine the dynamics of such a system.

For a system of N coupled 1-D oscillators there exist N "normal modes" in which all oscillators move with the same frequency and thus have fixed amplitude ratios (if each oscillator is allowed to move in α-D, then αN normal modes exist). The normal mode is for whole system. Even though uncoupled angular frequencies of the oscillators are not the same, the effect of coupling is that all bodies can move with the same frequency. If the initial state of the system corresponds to motion in a normal mode then the oscillations continue in the normal mode. However in general the motion is described by a linear combination of all the normal modes; since the differential equations are linear such a linear combination is also a solution to the coupled linear equations.

The existence and nature of normal modes is best illustrated by some examples so we first turn to the solution of coupled linear equations.

2 Solution of coupled linear differential equations with constant coefficients.

Consider a set of differential equations that are linear and have constant coefficients. The procedure for solving them is a minor extension of the procedure for solving a single linear differential equation with constant coefficients. The steps are:

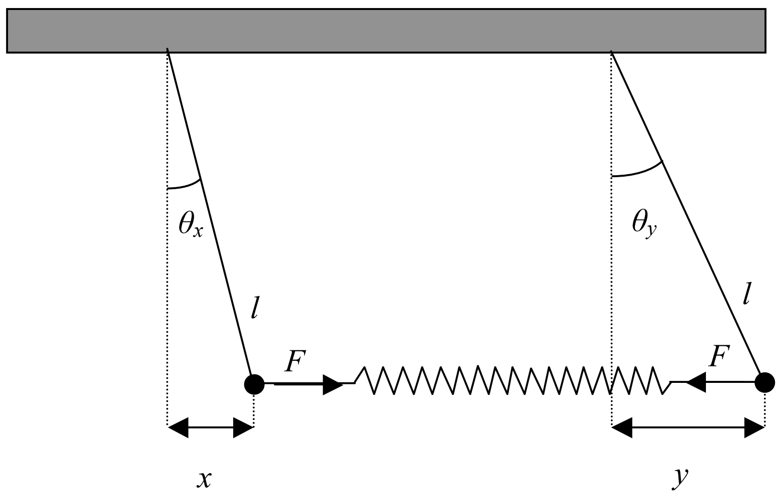
1. Arrange the equations so that terms on the left are all proportional to an unknown variable, and already known terms are on the right.

2. Find the general solution of the equations that are obtained by setting the right sides to zero. The result of this operation is the Complementary function (CF). For oscillatory solutions the CF is found by replacing the unknown variables by multiples of $e^{i\omega t}$ (if t is the independent variable) and solving the resulting algebraic equations.

3. Find a particular integral by putting in a trial solution for each term – polynomial, exponential, etc. – on the right hand side.

3 Coupled Pendula

The first example of coupled linear differential equations is provided by two coupled pendula. Consider two massless rods of length l, which have bobs of mass m attached to the end, which are themselves connected by a spring.



Assumptions:

1) Assume that spring obeys Hooke's law and thus that the restoring force varies linearly with extension, i.e. $F = k(y - x)$
2) Assume the displacements from equilibrium positions are small such that the restoring force due to gravity for each pendulum is approximately given by $mg\tan\theta = mg\dfrac{x}{l}$ and acts along the line of masses. The equations of motion are then:

$$m\ddot{x} = -mgx/l + k(y-x)$$
$$m\ddot{y} = -mgy/l - k(y-x) \qquad (3.1)$$

3.1 Matrix method of solution

We start with the general method of solution that applies to all coupled linear differential equations. As we will discuss there may be more direct methods in special cases. We first implement step 1 to write the equations of motion for the coupled pendula in a standard form

$$\begin{aligned} -\ddot{x}+\left(-\frac{g}{l}-\frac{k}{m}\right)x+\frac{k}{m}y=0 \\ \frac{k}{m}x-\ddot{y}+\left(-\frac{g}{l}-\frac{k}{m}\right)y=0 \end{aligned} \tag{3.2}$$

where the unknown variables are to the left. In this case there are no driving terms so the right hand side is zero. These equations may be written as a matrix equation

$$\begin{pmatrix} -\frac{d^2}{dt^2}-\left(\frac{g}{l}+\frac{k}{m}\right) & \frac{k}{m} \\ \frac{k}{m} & -\frac{d^2}{dt^2}-\left(\frac{g}{l}+\frac{k}{m}\right) \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}=\begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{3.3}$$

or

$$\mathbf{Ax}=\mathbf{0} \tag{3.4}$$

where $\mathbf{x}$ is the column vector $\begin{pmatrix} x \\ y \end{pmatrix}$ and $\mathbf{A}$ the square symmetric matrix in Eq.(3.3). Since the RHS is zero we are only interested in finding the CF. We look for normal mode solutions where all elements oscillate with the same frequency. Particularly for cases in which both first and second order derivatives are present (as is the case for damped oscillators discussed below) it is best to solve the associated complex equation. Writing

$$\mathbf{x}=\mathrm{Re}\left(\mathbf{Y}\equiv\mathbf{X}e^{i\omega t}\right) \tag{3.5}$$

where

$$\mathbf{X}=\begin{pmatrix} X \\ Y \end{pmatrix} \tag{3.6}$$

and substituting into Eq. 13 we find, since the differential operators are real, the associated complex equation is given by

$$\mathbf{AY}=0 \tag{3.7}$$

where

$$\mathbf{A}=\begin{pmatrix} \omega^2-\left(\frac{g}{l}+\frac{k}{m}\right) & \frac{k}{m} \\ \frac{k}{m} & \omega^2-\left(\frac{g}{l}+\frac{k}{m}\right) \end{pmatrix} \tag{3.8}$$

or, equivalently, dividing by the factor $e^{i\omega t}$

$$\mathbf{AX}=0 \tag{3.9}$$

The solutions of Eq.(3.9) are either $\mathbf{X}=0$, which is not very interesting, or the determinant of the matrix $\mathbf{A}$ must be equal to zero. Hence

$$\left|\begin{pmatrix} \omega^2-\left(\frac{g}{l}+\frac{k}{m}\right) & \frac{k}{m} \\ \frac{k}{m} & \omega^2-\left(\frac{g}{l}+\frac{k}{m}\right) \end{pmatrix}\right| = 0 \qquad (3.10)$$

leading to the "eigenvalue equation":

$$\omega^2-\left(\frac{g}{l}+\frac{k}{m}\right) = \pm\frac{k}{m} \qquad (3.11)$$

From this we see that there are two normal mode frequencies, $\omega_{1,2}$, corresponding to the two independent solutions of the coupled differential equations, given by

$$\begin{aligned} \omega_1 &= \sqrt{g/l} \\ \omega_2 &= \sqrt{g/l+2k/m} \end{aligned} \qquad (3.12)$$

(the $\pm$ ambiguity associated with the square root gives rise to the same sinusoidal solutions and so is ignored here). To complete the solution we need to find the normal mode amplitudes. These are found by solving Eq.(3.9) for $\mathbf{X}$, substituting each of the normal mode frequencies in turn. For $\omega=\omega_1$ we have

$$\begin{pmatrix} +\frac{k}{m} & -\frac{k}{m} \\ -\frac{k}{m} & +\frac{k}{m} \end{pmatrix}\begin{pmatrix} X_1 \\ Y_1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \qquad (3.13)$$

This determines the ratio of X_1 to Y_1 giving

$$\begin{pmatrix} X_1 \\ Y_1 \end{pmatrix} = A_1 e^{i\phi_1}\begin{pmatrix} 1 \\ 1 \end{pmatrix} \qquad (3.14)$$

with A_1, ϕ_1 the real amplitude and phase. Similarly for $\omega=\omega_2$

$$\begin{pmatrix} -\frac{k}{m} & -\frac{k}{m} \\ -\frac{k}{m} & -\frac{k}{m} \end{pmatrix}\begin{pmatrix} X_2 \\ Y_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \qquad (3.15)$$

giving

$$\begin{pmatrix} X_2 \\ Y_2 \end{pmatrix} = A_2 e^{i\phi_2}\begin{pmatrix} 1 \\ -1 \end{pmatrix} \qquad (3.16)$$

Finally, since $\mathbf{x}=\mathrm{Re}\left(\mathbf{X}e^{i\omega t}\right)$ we have the two "normal mode" solutions

$$\mathbf{x}_{1,2} = \begin{pmatrix} 1 \\ \pm 1 \end{pmatrix} A_{1,2}\cos(\omega_{1,2}t+\phi_{1,2}) \qquad (3.17)$$

and hence, since the differential equations are linear, we can use the principle of superposition to write the general solution as a linear combination of the two normal mode solutions

$$\mathbf{x} = \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} A_1\cos\left(\omega_1 t+\phi_1\right) + \begin{pmatrix} 1 \\ -1 \end{pmatrix} A_2\cos\left(\omega_2 t+\phi_2\right) \qquad (3.18)$$

The advantage of the matrix method is its general applicability, and the ease with which it may be applied to systems with more than two normal modes. The advantage of using the complex exponential is only evident if there is a mixture of single and double derivatives as in the case of a damped pendulum discussed below. In the undamped case just discussed it would be equally simple to start with a normal mode trial solution proportional to $\cos(\omega t+\phi)$.

3.2 Alternative methods of solution: Normal coordinates or Decoupling

The equations of motion for the coupled pendula are given by Eq.(3.1), rewritten here for convenience

$$\begin{aligned} m\ddot{x} &= -mgx/l + k(y-x) \\ m\ddot{y} &= -mgy/l - k(y-x) \end{aligned} \tag{3.19}$$

For simple coupled oscillator systems it is often possible to find the normal modes directly by taking obvious linear combinations of the equations of motion to obtain decoupled differential equations. These may then be independently solved for a linear combination of the position variables, in this case *x* and *y*. If this can be done it considerable simplifies the solution. The coupled pendula just discussed provides a simple example of this. If we add Eqs.(3.19) we find:

$$m(\ddot{x}+\ddot{y}) = -\frac{mg}{l}(x+y) \tag{3.20}$$

or

$$\ddot{q}_1 = -\frac{g}{l}q_1 \tag{3.21}$$

where q_1 is a *normal coordinate* here equal to $q_1=(x+y)/\sqrt{2}$ (The normalisation factor $1/\sqrt{2}$ is chosen to give a standard form for the kinetic energy when expressed in terms of the normal modes – see Eq.(3.41)).

Eq.(3.21) describes simple harmonic motion which may be trivially solved to give:

$$q_1 = \sqrt{2}A_1\cos(\omega_1 t+\phi_1) \tag{3.22}$$

where $\omega_1=\sqrt{g/l}$ is the first normal frequency found earlier and we have chosen the integration constants to agree with those found using the matrix method.
Similarly, if we subtract Eqs.(3.19) we find:

$$m(\ddot{x}-\ddot{y}) = -\frac{mg}{l}(x-y) - 2k(x-y) \tag{3.23}$$

or

$$\ddot{q}_2 = -\left(\frac{g}{l}+\frac{2k}{m}\right)q_2 \tag{3.24}$$

where q_2 is another normal coordinate, equal to $q_1=(x-y)/\sqrt{2}$. Eq.(3.24) also describes simple harmonic motion and thus

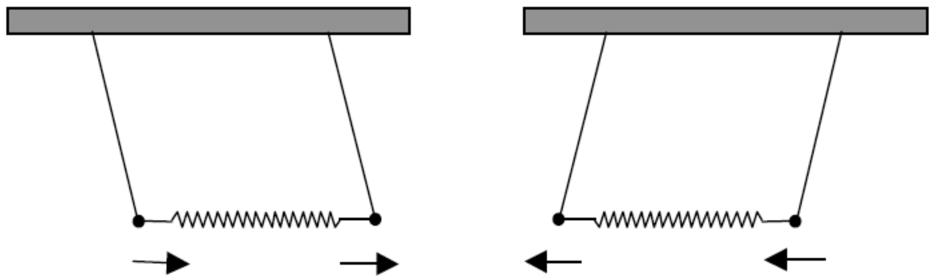
$$q_2 = \sqrt{2}A_2\cos(\omega_2 t+\phi_2) \tag{3.25}$$

where $\omega_2 = \sqrt{g/l + 2k/m}$ is the second normal frequency found earlier.

To extract the original position variables x and y we note that $x = (q_1 + q_2)/\sqrt{2}$ and $y = (q_1 - q_2)/\sqrt{2}$ and hence

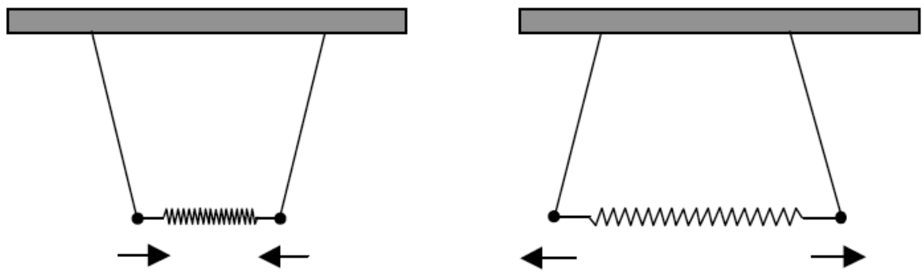
$$\begin{aligned} x(t) &= A_1 \cos(\omega_1 t + \phi_1) + A_2 \cos(\omega_2 t + \phi_2) \\ y(t) &= A_1 \cos(\omega_1 t + \phi_1) - A_2 \cos(\omega_2 t + \phi_2) \end{aligned} \tag{3.26}$$

which is identical to the general solutions Eq.(3.18) derived by the matrix method. From this is easy to identify the motion corresponding to the normal modes. For the case only the first normal mode is excited $A_2 = 0$ and the motion is shown in the figure below showing the two masses move together.



Normal mode 1

The second normal mode corresponds to the case $A_1 = 0$ and for it the masses move in opposite directions.



Normal Mode 2

This method of solution can lead to quick solutions for the normal frequencies if the suitable linear combination of parameters can be spotted. For simple cases like this it is easy but not for more complicated systems. This technique is also known as *decoupling.*

3.3 Initial conditions

The values of the integration constants A_i, ϕ_i are determined from the initial conditions of the system. As is shown in the following examples this can lead to a single normal mode being excited or to a combination of normal modes.

Example (a) – Normal Mode Excitation

Suppose that at $t = 0$, $x = a$, $y = a$ and the masses are initially at rest. Equating the initial positions to Eqs.(3.26) implies:

$$\begin{aligned} x(0) &= A_1 \cos\phi_1 + A_2 \cos\phi_2 = a \\ y(0) &= A_1 \cos\phi_1 - A_2 \cos\phi_2 = a \end{aligned} \tag{3.27}$$

which implies $A_1 \cos\phi_1 = a, A_2 \cos\phi_2 = 0$. Equating the initial velocities to zero gives

$$\begin{aligned} \dot{x}(0) &= A_1\omega_1 \sin\phi_1 + A_2\omega_2 \sin\phi_2 = 0 \\ \dot{y}(0) &= A_1\omega_1 \sin\phi_1 - A_2\omega_2 \sin\phi_2 = 0 \end{aligned} \tag{3.28}$$

giving $A_1 = a, A_2 = 0, \phi_1 = 0$. Hence the solution for $t > 0$ is

$$\begin{aligned} x &= a\cos\omega_1 t \\ y &= a\cos\omega_1 t \end{aligned} \tag{3.29}$$

and thus, c.f. Eq.(3.26), we see that only the first normal mode is excited, which is to be expected given the initial displacements. In addition, once in this normal mode, the system will remain in it indefinitely.

Example (b) – Normal Mode Excitation

Suppose that at $t = 0$, $x = y = 0$, and the masses are given initial velocities $\dot{x} = -\mathrm{v}$, $\dot{y} = \mathrm{v}$. This implies

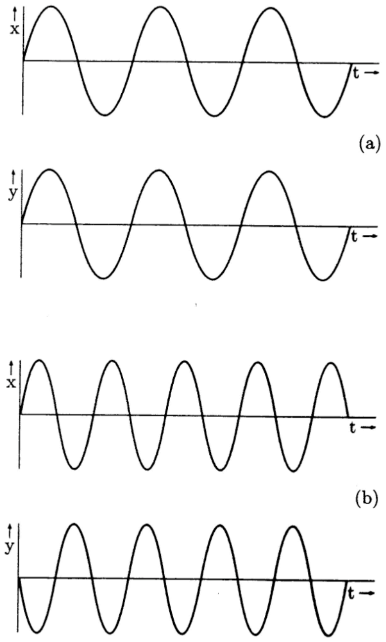
$$\begin{aligned} x(0) &= A_1 \cos\phi_1 + A_2 \cos\phi_2 = 0 \\ y(0) &= A_1 \cos\phi_1 - A_2 \cos\phi_2 = 0 \\ \dot{x}(0) &= A_1\omega_1 \sin\phi_1 + A_2\omega_2 \sin\phi_2 = -\mathrm{v} \\ \dot{y}(0) &= A_1\omega_1 \sin\phi_1 - A_2\omega_2 \sin\phi_2 = \mathrm{v} \end{aligned} \tag{3.30}$$

giving $A_1 = 0, A_2 = -\dfrac{v}{\omega_2}, \phi_2 = 0$ and thus the subsequent motion is:

$$\begin{aligned} x &= -\frac{v}{\omega_2}\sin\omega_2 t \\ y &= \frac{v}{\omega_2}\sin\omega_2 t \end{aligned} \tag{3.31}$$

Thus, c.f. Eq.(3.26), we see that these initial conditions excite the second normal mode only, in which the system will remain.

The motion in these two normal modes may also be summarised by the following figure:



where here the coupling is such that the frequency of the 2nd mode is higher than that of the first.

Example c – Non-Normal Behaviour – Beats

Suppose that at $t = 0$, $x = a$, $y = 0$ and the masses are initially at rest. This requires

$$\begin{aligned} x(0) &= A_1 \cos\phi_1 + A_2 \cos\phi_2 = a \\ y(0) &= A_1 \cos\phi_1 - A_2 \cos\phi_2 = 0 \\ \dot{x}(0) &= A_1\omega_1 \sin\phi_1 + A_2\omega_2 \sin\phi_2 = 0 \\ \dot{y}(0) &= A_1\omega_1 \sin\phi_1 - A_2\omega_2 \sin\phi_2 = 0 \end{aligned} \qquad (3.32)$$

giving $A_1 = A_2 = \frac{a}{2}, \phi_1 = \phi_2 = 0$. Hence the solution for $t > 0$ is

$$\begin{aligned} x &= \frac{a}{2}\cos\omega_1 t + \frac{a}{2}\cos\omega_2 t \\ y &= \frac{a}{2}\cos\omega_1 t - \frac{a}{2}\cos\omega_2 t \end{aligned} \qquad (3.33)$$

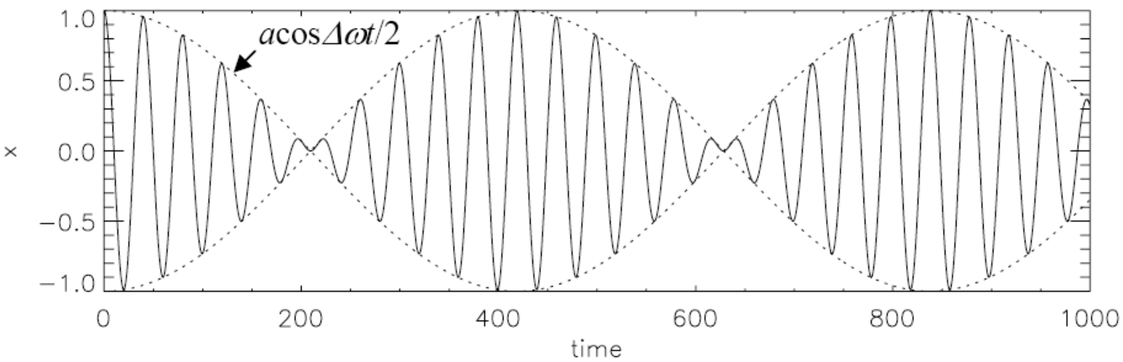
and thus both normal modes are excited. The solution for both x and y is then determined by the beating of the two terms with normal frequencies ω_1 and ω_2.

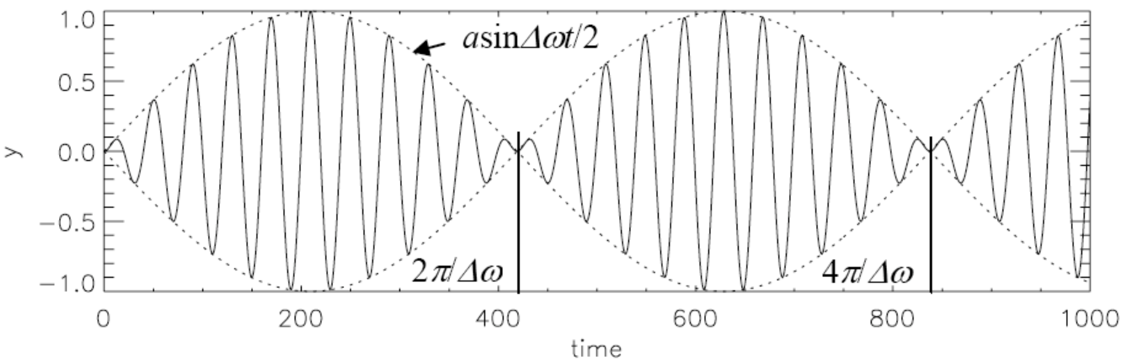
3.4 Beats

Eqs.(3.33) can be re-written using standard trigonometrical identities as:

$$x = a\cos\left(\frac{\omega_1 + \omega_2}{2}t\right)\cos\left(\frac{\omega_1 - \omega_2}{2}t\right) = a\cos\left(\bar{\omega}\, t\right)\cos\left(\Delta\omega\, t/2\right)$$
$$y = a\sin\left(\frac{\omega_1 + \omega_2}{2}t\right)\sin\left(\frac{\omega_1 - \omega_2}{2}t\right) = a\sin\left(\bar{\omega}\, t\right)\sin\left(\Delta\omega\, t/2\right) \qquad (3.34)$$

where $\bar{\omega} = \frac{\omega_1 + \omega_2}{2}$ and $\Delta\omega = \omega_1 - \omega_2$. The form of x and y is shown in the figures below for the case $\Delta\omega << \bar{\omega}$.





versa, corresponding to a transfer of energy between the two pendula. Note also that one complete period of the envelope equals <u>two</u> beats.

3.5 Energy of Motion

Decoupling, to express the result in terms of normal modes, is also instructive when the energy of the system is considered. Consider first the potential energy, $V(x,y)$, of the coupled oscillators. Consider the forces acting on particle 1 which, c.f. Eq.(3.19), are given by $-mgx/l + k\left(y - x\right)$. This force may be written in terms of a partial derivative with respect to x of a potential *V(x,y)*:

$$-mgx/l + k\left(y - x\right) = -\frac{\partial V}{\partial x} \qquad (3.35)$$

Integrating this we find:

$$V = \frac{mg}{2l}x^2 + \frac{1}{2}kx^2 - kxy + f\left(y\right) \qquad (3.36)$$

where *f* is an unknown function of *y*.

The force, $-mgy/l - k(y-x)$, acting on particle 2 may similarly be obtained from the potential energy giving

$$-mgy/l - k(y-x) = -\frac{\partial V}{\partial y} = kx - \frac{df}{dy} \tag{3.37}$$

where we have used Eq.(3.36) to compute the partial derivative. Integrating this equation determines $f(y)$ and inserted in Eq.(3.36) gives the total kinetic energy of the system (up to an undetermined constant)

$$V = \frac{1}{2}m\left(\frac{g}{l}+\frac{k}{m}\right)\left(x^2+y^2\right) - kxy \tag{3.38}$$

Consider now the kinetic energy *K*. This is given simply by:

$$K = \frac{1}{2}m\left(\dot{x}^2+\dot{y}^2\right) \tag{3.39}$$

While the expression for *K* is straightforward, that for *V* is rather more complex. However if we substitute for the normal coordinates: $x = (q_1+q_2)/\sqrt{2}$ and $y = (q_1-q_2)/\sqrt{2}$, then the *V* may be re-expressed as:

$$\begin{aligned} V &= \frac{1}{2}m\frac{g}{l}q_1^2 + \frac{1}{2}m\left(\frac{g}{l}+\frac{2k}{m}\right)q_2^2 \\ &= \frac{1}{2}m\omega_1^2q_1^2 + \frac{1}{2}m\omega_2^2q_2^2 \end{aligned} \tag{3.40}$$

where ω_1 and ω_2 are the normal mode angular frequencies. Similarly *K* may be rewritten as:

$$K = \frac{1}{2}m\left(\dot{q}_1^2+\dot{q}_2^2\right) \tag{3.41}$$

and the total energy is then:

$$\begin{aligned} E &= \frac{1}{2}m\omega_1^2q_1^2 + \frac{1}{2}m\omega_2^2q_2^2 + \frac{1}{2}m\dot{q}_1^2 + \frac{1}{2}m\dot{q}_2^2 \\ &= \left(\frac{1}{2}m\omega_1^2q_1^2 + \frac{1}{2}m\dot{q}_1^2\right) + \left(\frac{1}{2}m\omega_2^2q_2^2 + \frac{1}{2}m\dot{q}_2^2\right) \\ &= E_1 + E_2 = \sum_{1}^{N=2} E_n \end{aligned} \tag{3.42}$$

One may see from Eqs.(3.40),(3.41) and (3.42) that the energies separate into the individual energies of two decoupled simple harmonic oscillators corresponding to the motion of the two normal modes. The is an example of Parseval's theorem which states that **the total energy of the system is the sum of the energies of the normal modes.**

4 Coupled driven linear differential equations

We now consider examples in which there is a driving term forcing the motion, following the steps listed above when obtaining the solution. The examples appear in Section 3.8 of my lecture notes on Complex Numbers and Ordinary Differential Equations. For completeness I reproduce them here.

Example 3.22

Solve

$$\frac{\mathrm{d}x}{\mathrm{d}t} + \frac{\mathrm{d}y}{\mathrm{d}t} + y = t,$$
$$-\frac{\mathrm{d}y}{\mathrm{d}t} + 3x + 7y = \mathrm{e}^{2t} - 1.$$

It is helpful to introduce the shorthand

$$\mathbf{L}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \frac{\mathrm{d}x}{\mathrm{d}t} & +\frac{\mathrm{d}y}{\mathrm{d}t} + y \\ 3x & -\frac{\mathrm{d}y}{\mathrm{d}t} + 7y \end{pmatrix}.$$

To find CF

Set $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} X\mathrm{e}^{\alpha t} \\ Y\mathrm{e}^{\alpha t} \end{pmatrix}$ α, X, Y complex nos to be determined

Plug into $\mathbf{L}\begin{pmatrix} x \\ y \end{pmatrix} = 0$ and cancel the factor $\mathrm{e}^{\alpha t}$

$$\begin{aligned} \alpha X + (\alpha + 1)Y &= 0, \\ 3X + (7 - \alpha)Y &= 0. \end{aligned} \tag{3.48}$$

The theory of equations, to be discussed early next term, shows that these equations allow X, Y to be non-zero only if the determinant

$$\begin{vmatrix} \alpha & \alpha+1 \\ 3 & 7-\alpha \end{vmatrix} = 0,$$

which in turn implies that $\alpha(7-\alpha) - 3(\alpha+1) = 0 \quad \Rightarrow \quad \alpha = 3,\ \alpha = 1$. We can arrive at the same conclusion less quickly by using the second equation to eliminate Y from the first equation. So the bottom line is that $\alpha = 3, 1$ are the only two viable values of α. For each value of α either of equations (3.48) imposes a ratio* X/Y

$\alpha = 3 \quad \Rightarrow \quad 3X + 4Y = 0 \quad \Rightarrow \quad Y = -\frac{3}{4}X,$

$\alpha = 1 \quad \Rightarrow \quad X + 2Y = 0 \quad \Rightarrow \quad Y = -\frac{1}{2}X.$

Hence the CF made up of

$$\begin{pmatrix} x \\ y \end{pmatrix} = X_a \begin{pmatrix} 1 \\ -\frac{3}{4} \end{pmatrix} \mathrm{e}^{3t} \quad \text{and} \quad \begin{pmatrix} x \\ y \end{pmatrix} = X_b \begin{pmatrix} 1 \\ -\frac{1}{2} \end{pmatrix} \mathrm{e}^{t}.$$

The two arbitrary constants in this CF reflect the fact that the original equations were first-order in two variables.

To find PI

(i) *Polynomial part*

$$\text{Try} \quad \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} X_0 + X_1 t \\ Y_0 + Y_1 t \end{pmatrix}$$

Plug into $\mathbf{L}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} t \\ -1 \end{pmatrix}$

$$\begin{array}{cc} X_1 + Y_1 + Y_1 t + Y_0 = t & 3(X_0 + X_1 t) - Y_1 + 7(Y_0 + Y_1 t) = -1 \\ \Downarrow & \Downarrow \\ Y_1 = 1;\ X_1 + Y_1 + Y_0 = 0 & 3X_0 + 7Y_0 = 0;\ 3X_1 + 7Y_1 = 0 \\ \Downarrow & \Downarrow \\ X_1 + Y_0 = -1 & X_1 = -\frac{7}{3} \end{array}$$

Consequently, $Y_0 = -1 + \frac{7}{3} = \frac{4}{3}$ and $X_0 = -\frac{7}{3}Y_0 = -\frac{28}{9}$

Thus

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -\frac{28}{9} - \frac{7}{3}t \\ \frac{4}{3} + t \end{pmatrix}$$

(ii) *Exponential part*

$$\text{Try} \quad \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} X \\ Y \end{pmatrix} \mathrm{e}^{2t}$$

* The allowed values of α are precisely those for which you get the same value of X/Y from both of equations (3.48).

Plug into $\mathbf{L}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ \mathrm{e}^{2t} \end{pmatrix}$ and find

$$\begin{aligned} 2X + (2+1)Y = 0 \quad &\Rightarrow \quad X = -\tfrac{3}{2}Y \\ 3X + (-2+7)Y = 1 \quad &\Rightarrow \quad (-\tfrac{9}{2}+5)Y = 1 \end{aligned}$$

Hence $Y = 2$, $X = -3$.

Putting everything together the general solution is

$$\begin{pmatrix} x \\ y \end{pmatrix} = X_a \begin{pmatrix} 1 \\ -\frac{3}{4} \end{pmatrix} \mathrm{e}^{3t} + X_b \begin{pmatrix} 1 \\ -\frac{1}{2} \end{pmatrix} \mathrm{e}^{t} + \begin{pmatrix} -3 \\ 2 \end{pmatrix} \mathrm{e}^{2t} + \begin{pmatrix} -\frac{28}{9} - \frac{7}{3}t \\ \frac{4}{3} + t \end{pmatrix} \tag{3.49}$$

We can use the arbitrary constants in the above solution to obtain a solution in which x and y or $\dot{x}$ and $\dot{y}$ take on any prescribed values at $t = 0$:

Example 3.23

For the differential equations of Example 3.22, find the solution in which

$$\dot{x}(0) = -\tfrac{19}{3}$$
$$\dot{y}(0) = 3$$

Solution: Evaluate the time derivative of the GS at $t = 0$ and set the result equal to the given data:

$$\begin{pmatrix} -\frac{19}{3} \\ 3 \end{pmatrix} = 3X_a \begin{pmatrix} 1 \\ -\frac{3}{4} \end{pmatrix} + X_b \begin{pmatrix} 1 \\ -\frac{1}{2} \end{pmatrix} + 2 \begin{pmatrix} -3 \\ 2 \end{pmatrix} + \begin{pmatrix} -\frac{7}{3} \\ 1 \end{pmatrix}$$

Hence

$$\begin{aligned} 3X_a + X_b &= 2 \\ -\tfrac{9}{4}X_a - \tfrac{1}{2}X_b &= -2 \end{aligned} \quad \Rightarrow \quad \begin{aligned} X_a &= \frac{-2}{-3/2} = \tfrac{4}{3} \\ X_b &= 2 - 3X_a = -2 \end{aligned}$$

Here's another, more complicated example.

Example 3.24

Solve

$$\begin{aligned} \frac{\mathrm{d}^2 x}{\mathrm{d}t^2} + \frac{\mathrm{d}y}{\mathrm{d}t} + 2x &= 2\sin t + 3\cos t + 5\mathrm{e}^{-t} \\ \frac{\mathrm{d}x}{\mathrm{d}t} + \frac{\mathrm{d}^2 y}{\mathrm{d}t^2} - y &= 3\cos t - 5\sin t - \mathrm{e}^{-t} \end{aligned} \quad \text{given} \quad \begin{aligned} x(0) = 2; \quad y(0) = -3 \\ \dot{x}(0) = 0; \quad \dot{y}(0) = 4 \end{aligned}$$

To find CF

Set $x = X\mathrm{e}^{\alpha t}$, $y = Y\mathrm{e}^{\alpha t}$

$$\Rightarrow \quad \begin{matrix} (\alpha^2+2)X & + & \alpha Y \\ \alpha X & + & (\alpha^2-1)Y \end{matrix} = 0 \quad \Rightarrow \quad \alpha^4 = 2$$

$$\Rightarrow \quad \alpha^2 = \pm\sqrt{2} \quad \Rightarrow \quad \alpha = \pm\beta, \ \pm\mathrm{i}\beta \quad (\beta \equiv 2^{1/4})$$

and $Y/X = -(\alpha^2+2)/\alpha$ so the CF is

$$\begin{pmatrix} x \\ y \end{pmatrix} = X_a \begin{pmatrix} \beta \\ 2+\sqrt{2} \end{pmatrix} \mathrm{e}^{-\beta t} + X_b \begin{pmatrix} -\beta \\ 2+\sqrt{2} \end{pmatrix} \mathrm{e}^{\beta t}$$
$$+ X_c \begin{pmatrix} \mathrm{i}\beta \\ 2-\sqrt{2} \end{pmatrix} \mathrm{e}^{-\mathrm{i}\beta t} + X_d \begin{pmatrix} -\mathrm{i}\beta \\ 2-\sqrt{2} \end{pmatrix} \mathrm{e}^{\mathrm{i}\beta t}$$

Notice that the functions multiplying X_c and X_d are complex conjugates of one another. So if the solution is to be real X_d has to be the complex conjugate of X_c and these two complex coefficients contain only two real arbitrary constants between them. There are four arbitrary constants in the CF because we are solving second-order equations in two dependent variables.

To Find PI

Set $(x, y) = (X, Y)\mathrm{e}^{-t} \quad \Rightarrow$

$$\begin{matrix} X - Y + 2X = 5 \\ -X + Y - Y = -1 \end{matrix} \quad \Rightarrow \quad \begin{matrix} X = 1 \\ Y = -2 \end{matrix} \quad \Rightarrow \quad \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ -2 \end{pmatrix} \mathrm{e}^{-t}$$

Have $2\sin t + 3\cos t = \Re\mathrm{e}(\sqrt{13}\mathrm{e}^{\mathrm{i}(t+\phi)})$, where $\cos\phi = 3/\sqrt{13}$, $\sin\phi = -2/\sqrt{13}$.
Similarly $3\cos t - 5\sin t = \Re\mathrm{e}(\sqrt{34}\mathrm{e}^{\mathrm{i}(t+\psi)})$, where $\cos\psi = 3/\sqrt{34}$, $\sin\psi = 5/\sqrt{34}$
Set $(x, y) = \Re\mathrm{e}[(X, Y)\mathrm{e}^{\mathrm{i}t}]$ and require

$$\begin{matrix} -X + \mathrm{i}Y + 2X = X + \mathrm{i}Y = \sqrt{13}\mathrm{e}^{\mathrm{i}\phi} \\ \mathrm{i}X - Y - Y = \mathrm{i}X - 2Y = \sqrt{34}\mathrm{e}^{\mathrm{i}\psi} \end{matrix} \quad \Rightarrow \quad \begin{matrix} -\mathrm{i}Y = \sqrt{13}\mathrm{e}^{\mathrm{i}\phi} + \mathrm{i}\sqrt{34}\mathrm{e}^{\mathrm{i}\psi} \\ \mathrm{i}X = 2\mathrm{i}\sqrt{13}\mathrm{e}^{\mathrm{i}\phi} - \sqrt{34}\mathrm{e}^{\mathrm{i}\psi} \end{matrix}$$

so

$$\begin{aligned} x &= \Re\mathrm{e}(2\sqrt{13}\mathrm{e}^{\mathrm{i}(t+\phi)} + \mathrm{i}\sqrt{34}\mathrm{e}^{\mathrm{i}(t+\psi)}) \\ &= 2\sqrt{13}(\cos\phi\cos t - \sin\phi\sin t) - \sqrt{34}(\sin\psi\cos t + \cos\psi\sin t) \\ &= 2[3\cos t + 2\sin t] - 5\cos t - 3\sin t \\ &= \cos t + \sin t \end{aligned}$$

Similarly

$$\begin{aligned} y &= \Re\mathrm{e}(\sqrt{13}\mathrm{i}\mathrm{e}^{\mathrm{i}(t+\phi)} - \sqrt{34}\mathrm{e}^{\mathrm{i}(t+\psi)}) \\ &= \sqrt{13}(-\sin\phi\cos t - \cos\phi\sin t) - \sqrt{34}(\cos\psi\cos t - \sin\psi\sin t) \\ &= 2\cos t - 3\sin t - 3\cos t + 5\sin t \\ &= -\cos t + 2\sin t. \end{aligned}$$

Thus the complete PI is

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \cos t + \sin t \\ -\cos t + 2\sin t \end{pmatrix} + \begin{pmatrix} 1 \\ -2 \end{pmatrix} \mathrm{e}^{-t}.$$

For the initial-value problem

$$\mathrm{PI}(0) = \begin{pmatrix} 1 \\ -2 \end{pmatrix} + \begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 2 \\ -3 \end{pmatrix} \quad ; \quad \dot{\mathrm{PI}}(0) = \begin{pmatrix} -1 \\ 2 \end{pmatrix} + \begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 0 \\ 4 \end{pmatrix}$$
$$\mathrm{CF}(0) = \begin{pmatrix} 2 \\ -3 \end{pmatrix} - \begin{pmatrix} 2 \\ -3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad ; \quad \dot{\mathrm{CF}}(0) = \begin{pmatrix} 0 \\ 4 \end{pmatrix} - \begin{pmatrix} 0 \\ 4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

So the PI satisfies the initial data and $X_a = X_b = X_c = X_d = 0$.

In general the number of arbitrary constants in the general solution should be the sum of the orders of the highest derivative in each variable. There are exceptions to this rule, however, as the following example shows. This example also illustrates another general point: that before solving the given equations, one should always try to simplify them by adding a multiple of one equation or its derivative to the other.

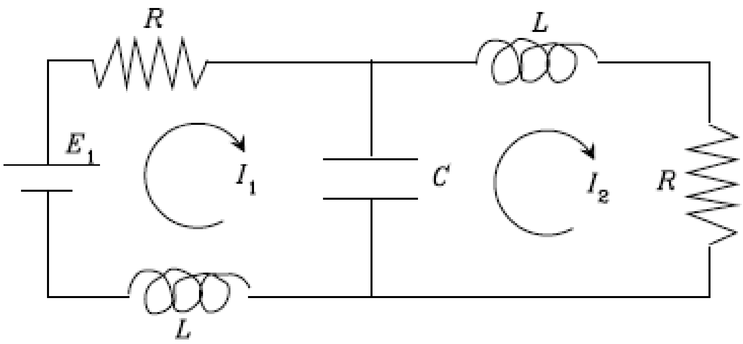
Example 3.25

Solve

$$\begin{aligned} \frac{\mathrm{d}x}{\mathrm{d}t} + \frac{\mathrm{d}y}{\mathrm{d}t} + y = t, \\ \frac{\mathrm{d}^2x}{\mathrm{d}t^2} + \frac{\mathrm{d}^2y}{\mathrm{d}t^2} + 3x + 7y = \mathrm{e}^{2t}. \end{aligned} \tag{3.50}$$

We differentiate the first equation and substract the result from the seond. Then the system becomes first-order – in fact the system solved in Example 3.22. From (3.49) we see that the general solution contains only two arbitrary constants rather than the four we might have expected from a cursory glance at (3.50). To understand this phenomenon better, rewrite the equations in terms of $z \equiv x + y$ as $\dot{z} + z - x = t$ and $\ddot{z} + 7z - 4x = \mathrm{e}^{2t}$. The first equation can be used to make x a function $x(z, \dot{z}, t)$. Using this to eliminate x from the second equation we obtain an expression for $\ddot{z}(z, \dot{z}, t)$. From this expression and its derivatives w.r.t. t we can construct a Taylor series for z once we are told $z(t_0)$ and $\dot{z}(t_0)$. Hence the general solution should have just two arbitrary constants.

3.8.1 LCR circuits The dynamics of a linear electrical circuit is governed by a system of linear equations with constant coefficients. These may be solved by the general technique described at the start of Chapter 4. In many cases they may be more easily solved by judicious addition and subtraction along the lines illustrated in Example 3.25.



Using Kirchhoff's laws

$$\begin{aligned} RI_1 + \frac{Q}{C} + L\frac{\mathrm{d}I_1}{\mathrm{d}t} &= E_1 \\ L\frac{\mathrm{d}I_2}{\mathrm{d}t} + RI_2 - \frac{Q}{C} &= 0. \end{aligned} \tag{3.51}$$

We first differentiate to eliminate Q

$$\begin{aligned} \frac{\mathrm{d}^2 I_1}{\mathrm{d}t^2} + \frac{R}{L}\frac{\mathrm{d}I_1}{\mathrm{d}t} + \frac{1}{LC}(I_1 - I_2) &= 0 \\ \frac{\mathrm{d}^2 I_2}{\mathrm{d}t^2} + \frac{R}{L}\frac{\mathrm{d}I_2}{\mathrm{d}t} - \frac{1}{LC}(I_1 - I_2) &= 0. \end{aligned} \tag{3.52}$$

We now add the equations to obtain

$$\frac{\mathrm{d}^2 S}{\mathrm{d}t^2} + \frac{R}{L}\frac{\mathrm{d}S}{\mathrm{d}t} = 0 \quad \text{where} \quad S \equiv I_1 + I_2. \tag{3.53}$$

Subtracting the equations we find

$$\frac{\mathrm{d}^2 D}{\mathrm{d}t^2} + \frac{R}{L}\frac{\mathrm{d}D}{\mathrm{d}t} + \frac{2}{LC}D = 0 \quad \text{where} \quad D \equiv I_1 - I_2. \tag{3.54}$$

We now have two uncoupled equations, one for S and one for D. We solve each in the standard way (Section 3.7).

3.8.2 Time evolution of the LCR circuits The auxiliary equation for (3.53) is $\alpha^2 + R\alpha/L = 0$, and its roots are

$$\alpha = 0 \quad \Rightarrow \quad S = \text{constant} \quad \text{and} \quad \alpha = -R/L \quad \Rightarrow \quad S \propto \mathrm{e}^{-Rt/L}. \tag{3.55}$$

Since the right side of (3.53) is zero, no PI is required.

The auxiliary equation for (3.54) is

$$\alpha^2 + \frac{R}{L}\alpha + \frac{2}{LC} = 0 \quad \Rightarrow \quad \alpha = -\tfrac{1}{2}\frac{R}{L} \pm \frac{\mathrm{i}}{\sqrt{LC}}\sqrt{2 - \tfrac{1}{4}CR^2/L} = -\tfrac{1}{2}\frac{R}{L} \pm \mathrm{i}\omega_R. \tag{3.56}$$

Again no PI is required.

Adding the results of (3.55) and (3.56), the general solutions to (3.53) and (3.54) are

$$I_1 + I_2 = S = S_0 + S_1\mathrm{e}^{-Rt/L} \quad ; \quad I_1 - I_2 = D = D_0\mathrm{e}^{-Rt/2L}\sin(\omega_R t + \phi).$$

From the original equations (3.52) it is easy to see that the steady-state currents are $I_1 = I_2 = \frac{1}{2}S_0 = \frac{1}{2}E_1/R$. Hence, the final general solution is

$$\begin{aligned} I_1 + I_2 &= S(t) = K\mathrm{e}^{-Rt/L} + \frac{E_1}{R} \\ I_1 - I_2 &= D(t) = D_0\mathrm{e}^{-Rt/2L}\sin(\omega_R t + \phi). \end{aligned} \tag{3.57}$$

Example 3.26

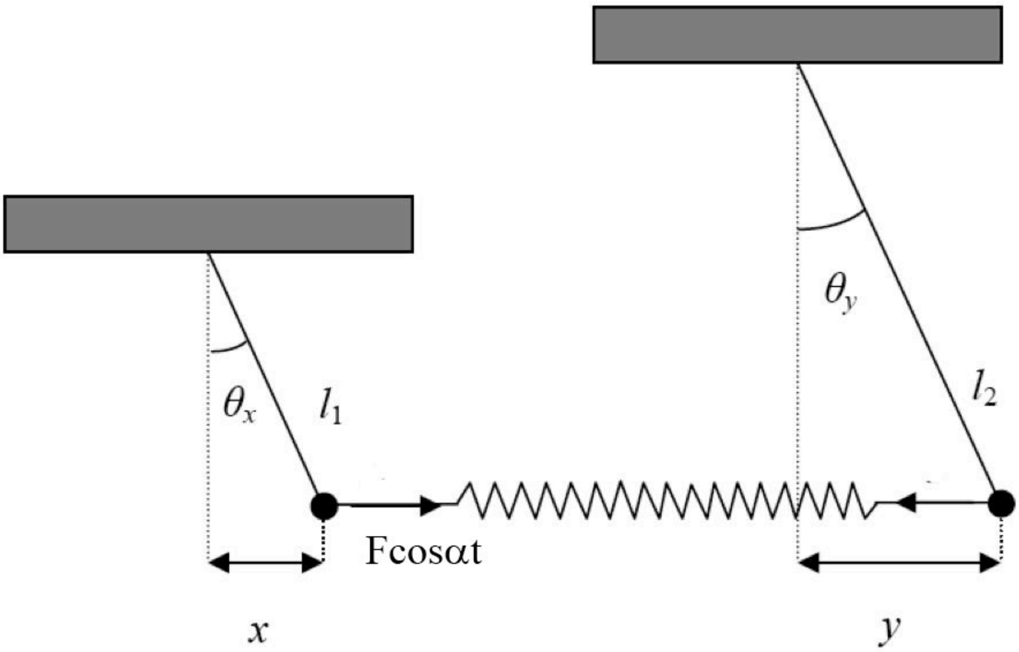
The battery is first connected up at $t = 0$. Determine I_1, I_2 for $t > 0$.

Solution: We have $I_1(0) = I_2(0) = 0$ and from the diagram we see that $\dot{I}_1(0) = E_1/L$ and $\dot{I}_2 = 0$. Looking at equations (3.57) we set $K = -E_1/R$ to ensure that $I_1(0) + I_2(0) = 0$, and $\phi = 0$ to ensure that $I_1(0) = I_2(0)$. Finally we set $D_0 = \dfrac{E_1}{L\omega_R}$ to ensure that $\dot{D}(0) = \dfrac{E_1}{L}$

5 Non-Identical Pendula, damped and with forced oscillations

Here we return to the coupled oscillator system to demonstrate how the matrix method provides a solution to the general case when the decoupling method is not straightforward to implement.

Consider the non-identical coupled oscillator system below with a force $F\cos\alpha t$ *acting on particle 1. Both* bobs are also subject to a frictional force equal to γ times their velocity.



The equations of motion are:

$$\begin{aligned} m_1\ddot{x} &= -\gamma\dot{x} - m_1 g x / l_1 + k(y-x) + F\cos\alpha t \\ m_2\ddot{y} &= -\gamma\dot{y} - m_2 g y / l_2 - k(y-x) \end{aligned} \quad (5.1)$$

This is equivalent to the matrix equation

$$\begin{pmatrix} \frac{d^2}{dt^2} + \frac{\gamma}{m_1}\frac{d}{dt} + \left(\frac{g}{l_1} + \frac{k}{m_1}\right) & -\frac{k}{m_1} \\ -\frac{k}{m_2} & \frac{d^2}{dt^2} + \frac{\gamma}{m_2}\frac{d}{dt} + \left(\frac{g}{l_2} + \frac{k}{m_2}\right) \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \frac{F}{m_1}\begin{pmatrix} 1 \\ 0 \end{pmatrix} \mathrm{Re}\left(e^{i\alpha t}\right) \quad (5.2)$$

The Complementary Function

The CF is found solving the equation with no driving term on the RHS. We look for a normal mode solution of the form

$$\begin{pmatrix} x \\ y \end{pmatrix} = \mathrm{Re}\left(\begin{pmatrix} X \\ Y \end{pmatrix} e^{i\omega t}\right) \tag{5.3}$$

Substituting this in the LHS of the matrix equation leads to the associated complex matrix equation

$$\begin{pmatrix} -\omega^2 + i\dfrac{\gamma}{m_1}\omega + \left(\dfrac{g}{l_1} + \dfrac{k}{m_1}\right) & -\dfrac{k}{m_1} \\ -\dfrac{k}{m_2} & -\omega^2 + i\dfrac{\gamma}{m_2}\omega + \left(\dfrac{g}{l_2} + \dfrac{k}{m_2}\right) \end{pmatrix} \begin{pmatrix} X \\ Y \end{pmatrix} = 0 \tag{5.4}$$

The associated eigenvalue equation is

$$\begin{vmatrix} -\omega^2 + i\dfrac{\gamma}{m_1}\omega + \left(\dfrac{g}{l_1} + \dfrac{k}{m_1}\right) & -\dfrac{k}{m_1} \\ -\dfrac{k}{m_2} & -\omega^2 + i\dfrac{\gamma}{m_2}\omega + \left(\dfrac{g}{l_2} + \dfrac{k}{m_2}\right) \end{vmatrix} = 0 \tag{5.5}$$

Solving this equation will give the normal mode frequencies. Finally substituting these frequencies in turn in Eq.(5.4) determines the normal modes in the usual manner. For the case of arbitrary masses and pendula lengths this matrix method is the optimal one to find the normal frequencies as it is not possible simply to identify the normal co-ordinates and apply the decoupling method.

The Particular Integral

To find the particular integral we try

$$\begin{pmatrix} x \\ y \end{pmatrix} = \mathrm{Re}\left(\begin{pmatrix} P \\ Q \end{pmatrix} e^{i\alpha t}\right) \tag{5.6}$$

Substituting this in the matrix equation the associated complex equation is

$$\begin{pmatrix} -\alpha^2 + i\dfrac{\gamma}{m_1}\alpha + \left(\dfrac{g}{l_1} + \dfrac{k}{m_1}\right) & -\dfrac{k}{m_1} \\ -\dfrac{k}{m_2} & -\alpha^2 + i\dfrac{\gamma}{m_2}\alpha + \left(\dfrac{g}{l_2} + \dfrac{k}{m_2}\right) \end{pmatrix} \begin{pmatrix} P \\ Q \end{pmatrix} \equiv \mathbf{MP} = \frac{F}{m_1}\begin{pmatrix} 1 \\ 0 \end{pmatrix} \tag{5.7}$$

where the factor $e^{i\alpha t}$ has been divided out of both sides. The solution to this equation is given by

$$\mathbf{P} \equiv \begin{pmatrix} P \\ Q \end{pmatrix} = \mathbf{M}^{-1}\frac{F}{m_1}\begin{pmatrix} 1 \\ 0 \end{pmatrix} \tag{5.8}$$

where

$$\mathbf{M} = \begin{pmatrix} -\alpha^2 + i\dfrac{\gamma}{m_1}\alpha + \left(\dfrac{g}{l_1} + \dfrac{k}{m_1}\right) & -\dfrac{k}{m_1} \\ -\dfrac{k}{m_2} & -\alpha^2 + i\dfrac{\gamma}{m_2}\alpha + \left(\dfrac{g}{l_2} + \dfrac{k}{m_2}\right) \end{pmatrix} \tag{5.9}$$

Finally the PI is given by

$$\begin{pmatrix} x \\ y \end{pmatrix} = \mathrm{Re}\left(\mathbf{M}^{-1} \frac{F}{m_1} \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{i\omega t} \right) \tag{5.10}$$

This illustrates how the matrix method may be used to obtain the solution to the general driven coupled pendula system. However evaluating the solution is algebraically complicated so to illustrate the final steps we consider a relatively simple case.

5.1 The case $m_1 = m_2 = m, \quad l_1 = l_2 = l$ - Matrix method

CF

In this case the eigenvalue equation, Eq.(5.5), becomes

$$\begin{aligned} &\left(-\omega^2 + i\frac{\gamma}{m}\omega + \left(\frac{g}{l}\right)\right)\left(-\omega^2 + i\frac{\gamma}{m}\omega + \left(\frac{g}{l} + \frac{2k}{m}\right)\right) \\ &\equiv \left(-\omega^2 + \omega_1^2 + i\frac{\gamma}{m}\omega \right)\left(-\omega^2 + \omega_2^2 + i\frac{\gamma}{m}\omega \right) = 0 \end{aligned} \tag{5.11}$$

with solutions

$$\bar{\omega}_{1,2} = i\frac{\gamma}{2m} \pm \sqrt{\omega_{1,2}^2 - \left(\frac{\gamma}{2m}\right)^2} \tag{5.12}$$

where $\omega_{1,2}$ are the normal frequencies for the case with no damping, c.f. Eq.(3.12).
For the case $\omega = \bar{\omega}_1$, corresponding to the first factor in Eq.(5.11) vanishing, ,$\left(-\bar{\omega}_1^{\,2} + i\frac{\gamma}{m}\bar{\omega}_1 + \frac{g}{l} \right) = 0$ the eigenvector equation becomes

$$\begin{pmatrix} \frac{k}{m} & -\frac{k}{m} \\ -\frac{k}{m} & \frac{k}{m} \end{pmatrix} \begin{pmatrix} X \\ Y \end{pmatrix} = 0 \tag{5.13}$$

implying

$$\begin{pmatrix} X \\ Y \end{pmatrix} = A_1 e^{i\phi_1} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \tag{5.14}$$

Similarly one readily finds the case $\omega = \bar{\omega}_2$ has its eigenvalues given by

$$\begin{pmatrix} X \\ Y \end{pmatrix} = A_2 e^{i\phi_2} \begin{pmatrix} 1 \\ -1 \end{pmatrix} \tag{5.15}$$

Putting this all together we have the complementary function

$$\begin{aligned} \begin{pmatrix} x \\ y \end{pmatrix} &= \mathrm{Re}\left(A_1 e^{i\phi_1} \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{-\frac{\gamma}{2m}t} e^{i\sqrt{\omega_1^2 - \left(\frac{\gamma}{2m}\right)^2}} \right) + \mathrm{Re}\left(A_2 e^{i\phi_2} \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{-\frac{\gamma}{2m}t} e^{i\sqrt{\omega_2^2 - \left(\frac{\gamma}{2m}\right)^2}} \right) \\ &= e^{-\frac{\gamma}{2m}t}\left(A_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} \cos\left(\sqrt{\omega_1^2 - \left(\frac{\gamma}{2m}\right)^2} + \phi_1 \right) + A_2 \begin{pmatrix} 1 \\ -1 \end{pmatrix} \cos\left(\sqrt{\omega_2^2 - \left(\frac{\gamma}{2m}\right)^2} + \phi_2 \right) \right) \end{aligned} \tag{5.16}$$

PI

For this choice of masses and lengths the matrix given in Eq.(5.9) is now

$$\mathbf{M} = \begin{pmatrix} -\alpha^2 + i\dfrac{\gamma}{m}\alpha + \left(\dfrac{g}{l} + \dfrac{k}{m}\right) & -\dfrac{k}{m} \\ -\dfrac{k}{m} & -\alpha^2 + i\dfrac{\gamma}{m}\alpha + \left(\dfrac{g}{l} + \dfrac{k}{m}\right) \end{pmatrix} \tag{5.17}$$

with inverse

$$\mathbf{M}^{-1} = \frac{1}{Det\,\mathbf{M}} \begin{pmatrix} -\alpha^2 + i\dfrac{\gamma}{m}\alpha + \left(\dfrac{g}{l} + \dfrac{k}{m}\right) & +\dfrac{k}{m} \\ +\dfrac{k}{m} & -\alpha^2 + i\dfrac{\gamma}{m}\alpha + \left(\dfrac{g}{l} + \dfrac{k}{m}\right) \end{pmatrix} \tag{5.18}$$

where

$$Det\,\mathbf{M} = \left(-\alpha^2 + i\frac{\gamma}{m}\alpha + \left(\frac{g}{l}\right)\right)\left(-\alpha^2 + i\frac{\gamma}{m}\alpha + \left(\frac{g}{l} + \frac{2k}{m}\right)\right) \tag{5.19}$$

It is convenient to rewrite this in the form

$$Det\,\mathbf{M} = B_1 e^{-i\theta_1}.B_2 e^{-i\theta_2} \tag{5.20}$$

where

$$\begin{aligned} B_{1,2} &= \left(\left(-\alpha^2 + \omega_{1,2}^2\right)^2 + \left(\frac{\alpha\gamma}{m}\right)^2\right)^{1/2} \\ \tan\theta_{1,2} &= \frac{-\alpha\gamma / m}{\left(-\alpha^2 + \omega_{1,2}^2\right)^2} \end{aligned} \tag{5.21}$$

Then $\mathbf{M}^{-1}$ may be rewritten as

$$\mathbf{M}^{-1} = \frac{e^{i(\theta_1+\theta_2)}}{2B_1B_2}\begin{pmatrix} B_1e^{-i\theta_1} + B_2e^{-i\theta_2} & -B_1e^{-i\theta_1} + B_2e^{-i\theta_2} \\ -B_1e^{-i\theta_1} + B_2e^{-i\theta_2} & B_1e^{-i\theta_1} + B_2e^{-i\theta_2} \end{pmatrix} \tag{5.22}$$

so finally, using Eq.(5.10) we have

$$\begin{aligned} \begin{pmatrix} x \\ y \end{pmatrix} &= \mathrm{Re}\left[\frac{F}{2m}\frac{e^{i(\theta_1+\theta_2)}}{B_1B_2}e^{i\alpha t}\begin{pmatrix} B_1e^{-i\theta_1} + B_2e^{-i\theta_2} \\ -B_1e^{-i\theta_1} + B_2e^{-i\theta_2} \end{pmatrix}\right] \\ &= \frac{F}{2mB_1B_2}\begin{pmatrix} B_1\cos(\alpha t + \theta_2) + B_2\cos(\alpha t + \theta_1) \\ -B_1\cos(\alpha t + \theta_2) + B_2\cos(\alpha t + \theta_1) \end{pmatrix} \end{aligned} \tag{5.23}$$

5.2 The case $m_1 = m_2 = m, \quad l_1 = l_2 = l$ - Decoupling method

For this choice of masses and lengths the decoupling method provides another way of identifying the normal modes and decoupling the differential equations for the driven oscillators. It is instructive to compare this to the matrix method. The coupled differential equations are

$$\begin{aligned} m_1\ddot{x} &= -\gamma\dot{x} - m_1 gx / l_1 + k\left(y - x\right) + F\cos\alpha t \\ m_2\ddot{y} &= -\gamma\dot{y} - m_2 gy / l_2 - k\left(y - x\right) \end{aligned} \tag{5.24}$$

The first normal mode

Adding the equations gives

$$\left(\ddot{x} + \ddot{y}\right) = -\frac{g}{l}\left(x + y\right) - \frac{\gamma}{m}\left(\dot{x} + \dot{y}\right) + \frac{F}{m}\cos\alpha t \tag{5.25}$$

or

$$\ddot{q}_1 + \frac{\gamma}{m}\dot{q}_1 + \frac{g}{l}q_1 = \frac{F}{\sqrt{2}m}\cos\alpha t = \frac{F}{\sqrt{2}m}\mathrm{Re}\, e^{i\alpha t} \tag{5.26}$$

CF

The auxiliary equation is

$$-\omega^2 + i\frac{\gamma}{m}\omega + \frac{g}{l} = 0 \tag{5.27}$$

with eigenvalues

$$\overline{\omega}_{1,2} = i\frac{\gamma}{2m} \pm \sqrt{\omega_1^2 - \left(\frac{\gamma}{2m}\right)^2} \tag{5.28}$$

as we found using the matrix method. The CF is then

$$q_1 = \sqrt{2}A_1 e^{-\frac{\gamma}{2m}t}\cos\left(\sqrt{\omega_1^2 - \left(\frac{\gamma}{2m}\right)^2} + \delta_1\right) \tag{5.29}$$

where $q_1 = \frac{1}{\sqrt{2}}(x + y)$.

PI

To find the P.I. put $q_1 = \mathrm{Re}\left[C_1\exp\left(i\alpha t\right)\right]$ then, c.f. Eq.(5.21):

$$\left(-\alpha^2 + i\alpha\frac{\gamma}{m} + \frac{g}{l}\right)C_1 \equiv B_1 e^{-i\theta_1}C_1 = \frac{F}{\sqrt{2}m} \tag{5.30}$$

and thus

$$C_1 = \frac{F}{\sqrt{2}mB_1}\exp\left(i\theta_1\right) \tag{5.31}$$

Hence the PI for the normal coordinate q_1 is given by:

$$q_1 = \frac{F}{\sqrt{2}mB_1}\cos(\alpha t + \theta_1) \tag{5.32}$$

The second normal mode

Subtracting the equations of motion (Eq.(5.24)) gives:

$$(\ddot{x} - \ddot{y}) + \frac{\gamma}{m}(\dot{x} - \dot{y}) + \frac{g}{l}(x - y) + \frac{2k}{m}(x - y) = \frac{F}{m}\cos\alpha t \tag{5.33}$$

or

$$\ddot{q}_2 + \frac{\gamma}{m}\dot{q}_2 + \left(\frac{g}{l} + \frac{2k}{m}\right)q_2 = \frac{F}{\sqrt{2}m}\cos\alpha t = \mathrm{Re}\left[\frac{F}{\sqrt{2}m}\exp(i\alpha t)\right] \tag{5.34}$$

where $q_2 = \frac{1}{\sqrt{2}}(x - y)$

CF

In a similar manner we readily find that the complementary function is given by

$$q_2 = \sqrt{2}A_2 e^{-\frac{\gamma}{2m}t}\cos\left(\sqrt{\omega_2^2 - \left(\frac{\gamma}{2m}\right)^2} + \delta_2\right) \tag{5.35}$$

PI

To find the P.I. put $q_2 = \mathrm{Re}\left[C_2\exp(i\alpha t)\right]$ then:

$$\left(-\alpha^2 + i\alpha\frac{\gamma}{m} + \frac{g}{l} + \frac{2k}{m}\right)C_2 \equiv B_2 e^{i\theta_2}C_2 = \frac{F}{\sqrt{2}m} \tag{5.36}$$

and thus

$$C_2 == \frac{F}{\sqrt{2}mB_2}\exp(i\phi_2) \tag{5.37}$$

Hence the normal coordinate q_2 is given by:

$$q_2 = \frac{F}{\sqrt{2}mB_2}\cos(\alpha t + \phi_2) \tag{5.38}$$

It is easy to solve for x and y giving

$$x = A_1 e^{-\frac{\gamma}{2m}t}\cos\left(\sqrt{\omega_1^2-\left(\frac{\gamma}{2m}\right)^2}+\delta_1\right)+A_2 e^{-\frac{\gamma}{2m}t}\cos\left(\sqrt{\omega_2^2-\left(\frac{\gamma}{2m}\right)^2}+\delta_2\right)$$
$$+\frac{F}{2mB_1}\cos\left(\alpha t+\theta_1\right)+\frac{F}{2mB_2}\cos\left(\alpha t+\phi_2\right)$$
$$y = A_1 e^{-\frac{\gamma}{2m}t}\cos\left(\sqrt{\omega_1^2-\left(\frac{\gamma}{2m}\right)^2}+\delta_1\right)-A_2 e^{-\frac{\gamma}{2m}t}\cos\left(\sqrt{\omega_2^2-\left(\frac{\gamma}{2m}\right)^2}+\delta_2\right)$$
$$+\frac{F}{2mB_1}\cos\left(\alpha t+\theta_1\right)-\frac{F}{2mB_2}\cos\left(\alpha t+\phi_2\right) \tag{5.39}$$

in agreement with the result obtained by the matrix method, Eqs.(5.16) and (5.23)
.

5.3 The case $m_1 = m_2 = m, \quad l_1 \neq l_2$, no damping, no driving force.

The final example we shall consider is the case that the masses are equal and there is no driving force but the pendula lengths differ. Fron Eq.(5.5) the eigenvalue equation is

$$\begin{vmatrix} -\omega^2+\left(\frac{g}{l_1}+\frac{k}{m}\right) & -\frac{k}{m} \\ -\frac{k}{m} & -\omega^2+\left(\frac{g}{l_2}+\frac{k}{m}\right) \end{vmatrix}=0 \tag{5.40}$$

Putting:

$$\begin{aligned} A &= g/l_1 + k/m = \beta_1^2 + k/m \\ B &= -k/m \\ C &= g/l_2 + k/m = \beta_2^2 + k/m \end{aligned} \tag{5.41}$$

gives

$$\omega_{1,2}^2=\frac{1}{2}\left[\left(\beta_1^2+\beta_2^2\right)+2k/m\pm\sqrt{\left(\beta_1^2-\beta_2^2\right)^2+\left(2k/m\right)^2}\right] \tag{5.42}$$

Substituting $\omega_{1,2}$ in Eq.(5.4) determines the eigenvectors $\mathbf{X}=\begin{pmatrix}x_0\\y_0\end{pmatrix}$ up to an overall constant:

$$\frac{x_0}{y_0}=-\frac{m}{2k}\left[\left(\beta_1^2-\beta_2^2\right)\pm\sqrt{\left(\beta_1^2-\beta_2^2\right)^2+\left(2k/m\right)^2}\right] \tag{5.43}$$

N.B. $(x_0/y_0)_1$ for mode 1 and $(x_0/y_0)_2$ for mode 2 are related by:

$$\left(\frac{y_0}{x_0}\right)_1=-1/\left(\frac{y_0}{x_0}\right)_2\equiv r \tag{5.44}$$

The full solution is then given by

$$\mathbf{x}(t)=\begin{pmatrix}x\\y\end{pmatrix}=\begin{pmatrix}1\\r\end{pmatrix}D\cos(\omega_1 t+\delta_1)+\begin{pmatrix}-r\\1\end{pmatrix}G\cos(\omega_2 t+\delta_2) \qquad (5.45)$$

Suppose at t = 0, $\mathbf{x}(0)=\begin{pmatrix}x\\y\end{pmatrix}=\begin{pmatrix}a\\0\end{pmatrix}$ and $\dot{\mathbf{x}}=0$. Having zero initial velocities means that $\delta_1=\delta_2=0$. Hence

$$\mathbf{x}(0)=\begin{pmatrix}a\\0\end{pmatrix}=D\begin{pmatrix}1\\r\end{pmatrix}+G\begin{pmatrix}-r\\1\end{pmatrix} \qquad (5.46)$$

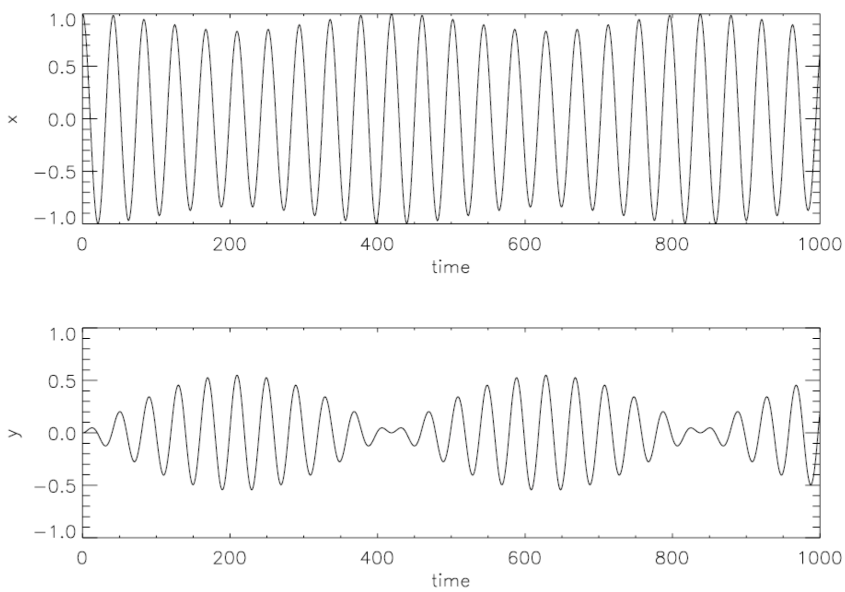
which may be solved to give

$$\begin{aligned}x(t)&=a\left[\cos\omega_1 t+r^2\cos\omega_2 t\right]/\left(1+r^2\right)\\ y(t)&=ar\left[\cos\omega_1 t-\cos\omega_2 t\right]/\left(1+r^2\right)\end{aligned} \qquad (5.47)$$

This is a little more difficult to simplify but it can be shown that

$$\begin{aligned}x(t)&=a\cos(\bar{\omega}t)\cos(\Delta\omega t/2)-a\left(\frac{1-r^2}{1+r^2}\right)\sin(\bar{\omega}t)\sin(\Delta\omega t/2)\\ y(t)&=2ar\sin(\bar{\omega}t)\sin(\Delta\omega t/2)/\left(1+r^2\right)\end{aligned} \qquad (5.48)$$

From this one sees that $|x|$ varies between a and $\left(\frac{1-r^2}{1+r^2}\right)a$ and $|y|$ varies between 0 and $\left(\frac{2r}{1+r^2}\right)a$. Hence, unlike the case for equal length pendula, there is an incomplete transfer of energy. This is clear from the plot of



$x(t)$ and $y(t)$:

Figure showing beats of non-identical pendula. Note the incomplete energy transfer.

5.4 Diagrammatic Representation of Normal Modes

The normal mode motion is specified by the ratio x_0/y_0. We can represent this by a unit-length vector $\mathbf{v} = \left(x_0\mathbf{i} + y_0\mathbf{j}\right)/\sqrt{x_0^2 + y_0^2}$. For the case of two normal modes there are two vectors.

Consider case of non-identical pendula discussed in Section 5.3. In various limits these eigenvalues defining the normal modes are given by

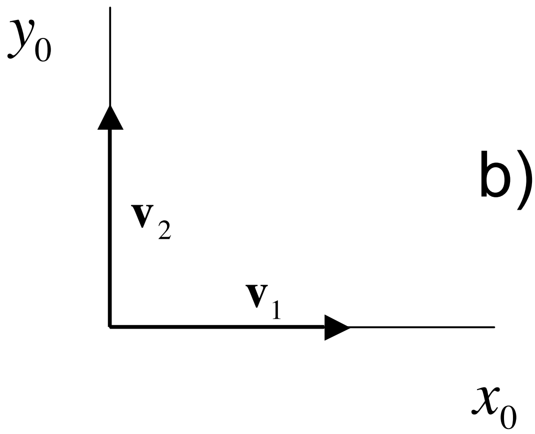
		$\mathbf{v}_1$		$\mathbf{v}_2$
(a) For $k/m \to 0$	$x_0/y_0. \to$	$-\infty$	or	0
(b) For $k/m \to \infty$	$x_0/y_0. \to$	-1	or	1

(c) Intermediate k/m

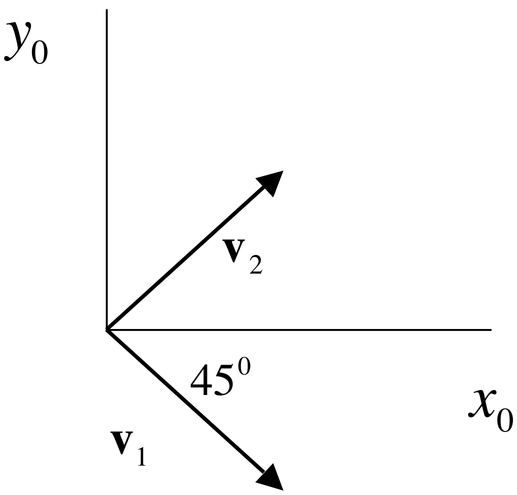
$$\frac{y_0}{x_0} = \tan\theta = \frac{-2k/m}{\left(\beta_1^2 - \beta_2^2\right) \pm \sqrt{\left(\beta_1^2 - \beta_2^2\right)^2 + \left(2k/m\right)^2}}$$

The corresponding graphical representation is given by

a).



b).



c).