NORMAL MODES AND WAVES

Hilary Term 2009.

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Question Sheet 1: Normal Modes

[Questions marked with an asterisk (*) cover topics also covered by the unstarred questions]

1. Two coupled simple pendula are of equal length l, but their bobs have different masses m_1 and m_2. Their equations of motion are:

$$\ddot{x} = -\frac{g}{l}x - \frac{k}{m_1}(x-y)$$

$$\text{and } \ddot{y} = -\frac{g}{l}y + \frac{k}{m_2}(x-y)$$

(a) Use the standard (matrix) method first to find the frequencies and the relative amplitudes of the bobs for the normal modes of the system.

(b) By taking suitable linear combinations of the two equations of motion, obtain two uncoupled differential equations for linear combinations of x and y. Hence again find the normal mode frequencies and the relative amplitudes. [Hint: One of these linear combinations is fairly obvious. For the other, it may be helpful to consider the centre of mass of the two bobs.]

Write down the most general solutions for x and y.

2. Consider again the two pendula of question 1. At $t = 0$, both pendula are at rest, with $x = A$ and $y = 0$. They are then released.

(a) Determine the subsequent motion of the system.

(b) Give two sets of initial conditions such that the subsequent motion of the pendula corresponds to each of the normal modes.

(c) For the case $m_1 = m_2 = m$, $\frac{k}{m} = 0.105\frac{g}{l}$:

- With the original initial conditions of part (a), show that

$$x = A\cos\Delta t\cos\bar{\omega}t$$

$$\text{and } y = A\sin\Delta t\sin\bar{\omega}t$$

where $\Delta = 0.05\sqrt{g/l}$ and $\bar{\omega} = 1.05\sqrt{g/l}$.

Sketch x and y, and note that the oscillations are transferred from the first pendulum to the second and back. Approximately how many oscillations does the second pendulum have before the first pendulum is oscillating again with its initial amplitude?

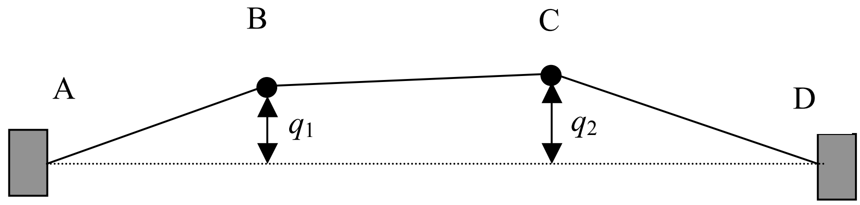
- At $t = 0$, both bobs are at their equilibrium positions: the first is stationary but the second is given an initial velocity v_0. Show that subsequently

$$x = \frac{v_0}{2}\left(\frac{1}{\omega_1}\sin\omega_1 t - \frac{1}{\omega_2}\sin\omega_2 t\right)$$

$$\text{and } y = \frac{v_0}{2}\left(\frac{1}{\omega_1}\sin\omega_1 t + \frac{1}{\omega_2}\sin\omega_2 t\right)$$

Describe as fully as possible the subsequent velocities of the two bobs.

3. The figure shows two masses m at points B and C of a string fixed at A and D, executing small transverse oscillations. The tensions are assumed to be all equal, and in equilibrium AB = BC = CD = l.



If the (small) transverse displacements of the masses are denoted by q_1 and q_2, the equations of motion are

$$m\ddot{q}_1 = -k\left(2q_1 - q_2\right), \qquad m\ddot{q}_2 = -k\left(2q_2 - q_1\right) \qquad (1)$$

where $k = T / l$, and terms of order q_1^2, q_2^2 and higher have been neglected.

(a) Define the *normal coordinates* Q_1, Q_2 by

$$Q_1 = \left(q_1 + q_2\right)/\sqrt{2}, \qquad Q_2 = \left(q_1 - q_2\right)/\sqrt{2}$$

Show that $m\ddot{Q}_1 = -kQ_1$, $m\ddot{Q}_2 = -3kQ_2$, and hence that the general solution of (1) is

$$Q_1 = E\cos\omega_1 t + F\sin\omega_1 t, \qquad Q_2 = G\cos\omega_2 t + H\sin\omega_2 t \qquad (2)$$

where $\omega_1 = \sqrt{k/m}$ and $\omega_2 = \sqrt{3k/m}$ are the *normal mode frequencies*. Hence find the general solution for q_1 and q_2.

(b) The forces on the RHS of (1) may be interpreted in terms of a potential energy function $V(q_1, q_2)$, as follows. We write the equations as

$$m\ddot{q}_1 = -\frac{\partial V}{\partial q_1}, \qquad m\ddot{q}_2 = -\frac{\partial V}{\partial q_2},$$

generalising "$m\ddot{x} = -\partial V / \partial x$". Show that V may be taken to be

$$V = k\left(q_1^2 + q_2^2 - q_1 q_2\right).$$

Derive the same result for V by considering the work done in giving each section of the string its deformation for equilibrium (e.g. for AB the work done is equal to $T\left(\sqrt{l^2+q_1^2}-l\right)$, and expand in powers of q_1^2/l^2 *only*.

Show that, when written in terms of the variables Q_1 and Q_2, V becomes

$$V=\frac{1}{2}m\omega_1^2Q_1^2+\frac{1}{2}m\omega_2^2Q_2^2$$

where $\omega_1=\sqrt{k/m}$ and $\omega_2=\sqrt{3k/m}$ as before.

(c) Show that the *kinetic energy* of the masses is

$$K=\frac{1}{2}m\left(\dot{Q}_1^2+\dot{Q}_1^2\right)$$

and hence that the total energy, in terms of Q_1 and Q_2, is

$$V+K=\left(\frac{1}{2}m\dot{Q}_1^2+\frac{1}{2}m\omega_1^2Q_1^2\right)+\left(\frac{1}{2}m\dot{Q}_2^2+\frac{1}{2}m\omega_2^2Q_2^2\right)=E_1+E_2$$

where E_1 is the total energy of 'oscillator' Q_1 with frequency ω_1, and similarly for E_2.

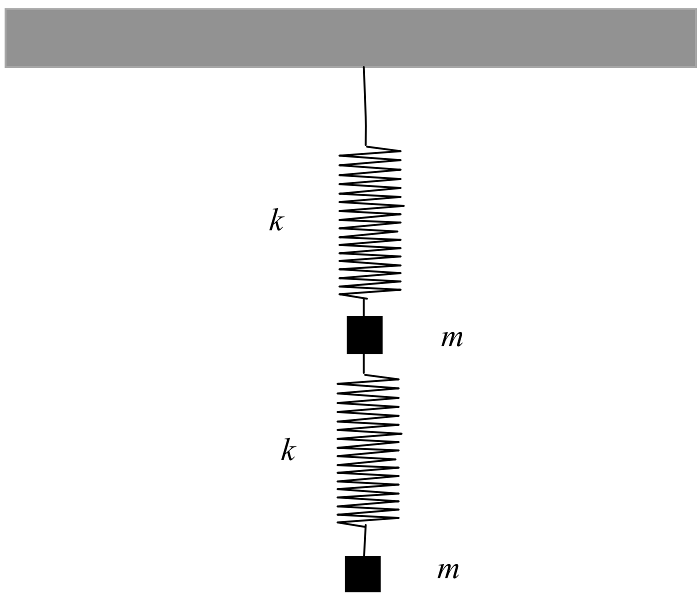
What is the expression for the total energy when written in terms of $q_1, q_2, \dot{q}_1$, and $\dot{q}_2$? Discuss the similarities and differences.

(d) Find the equations of motion for Q_1 and Q_2 from Newton's law in the form

$$m\ddot{Q}_1=-\frac{\partial V}{\partial Q_1}, \qquad m\ddot{Q}_2=-\frac{\partial V}{\partial Q_2},$$

and hence re-derive solution (2).

4.*

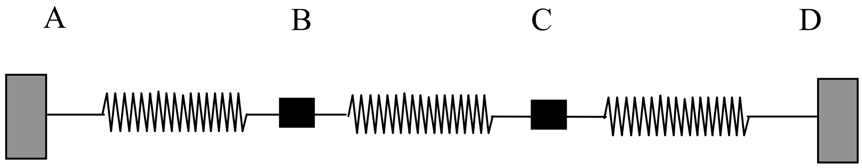


Two equal masses m are connected as shown with two identical massless springs, of spring constant k. Considering only motion in the vertical direction, obtain the differential equations for the displacements of the two masses from their equilibrium positions. Show that the angular frequencies of the normal modes are given by

$$\omega^2 = \left(3 \pm \sqrt{5}\right)k/2m$$

Find the ratio of the amplitudes of the two masses in each separate mode. Why does the acceleration due to gravity not appear in these answers?

5. *AB, BC, and CD are identical springs with negligible mass, and stiffness constant k:



The masses m, fixed to the springs at B and C, are displaced by small distances x_1 and x_2 from their equilibrium positions along the line of the springs, and execute small oscillations. Show that the angular frequencies of the normal modes are $\omega_1 = \sqrt{k/m}$ and $\omega_2 = \sqrt{3k/m}$. Sketch how the two masses move in each mode. Find x_1 and x_2 at times $t > 0$ if at $t = 0$ the system is at rest with $x_1 = a$, $x_2 = 0$.

6. *The setup is as for question 5, except that in this case the springs AB and CD have stiffness constant k_0, while BC has stiffness constant k_1. If C is clamped, B vibrates with frequency ν_0 = 1.81 Hz. The frequency of the lower frequency normal mode is ν_1 = 1.14 Hz. Calculate the frequency of the higher frequency normal mode, and the ratio k_1/k_0. (From French 5-7).

7. Two particles 1 and 2, each of mass m, are connected by a light spring of stiffness k, and are free to slide along a smooth horizontal track. What are the normal frequencies of this system? Describe the motion in the mode of zero frequency. Why does a zero-frequency mode appear in this problem, but not in question 5, for example?

   Particle 1 is now subject to a harmonic driving force $F\cos\omega t$. In the steady state, the amplitudes of vibration of 1 and 2 are A and B respectively. Find A and B, and discuss qualitatively the behaviour of the system as ω^2 is slowly increased from values near zero to values greater than $2k/m$.

8. A stretched massless string has its ends at $x = 0$ and $x = 3l$ fixed, and has equal masses attached at $x = l$ and $x = 2l$. Show that the equations of the transverse motion of the masses are approximately

$$m\ddot{y}_1 = \frac{T}{l}(y_2 - 2y_1)$$

$$\text{and } m\ddot{y}_2 = \frac{T}{l}(y_1 - 2y_2)$$

where T is the tension in the string. (Gravity does not appear in this equation because (i) y_1 and y_2 refer to motion with respect to the equilibrium positions; or (ii) the motion takes place on a horizontal frictionless table.) Convince yourself that, for small oscillations, it is reasonable to neglect the changes in tension caused by the variation in length of the three sections of the string resulting from the transverse motion of the masses.

Find the frequencies and the ratio of amplitudes of the transverse oscillations for the normal modes of the two masses. Is the relative motion of the higher-frequency mode reasonable?

9. The currents i_1 and i_2 in two coupled LC circuits satisfy the equations

$$L\frac{\mathrm{d}^2 i_1}{\mathrm{d}t^2} + \frac{i_1}{C} - M\frac{\mathrm{d}^2 i_2}{\mathrm{d}t^2} = 0$$
$$L\frac{\mathrm{d}^2 i_2}{\mathrm{d}t^2} + \frac{i_2}{C} - M\frac{\mathrm{d}^2 i_1}{\mathrm{d}t^2} = 0 \ ,$$

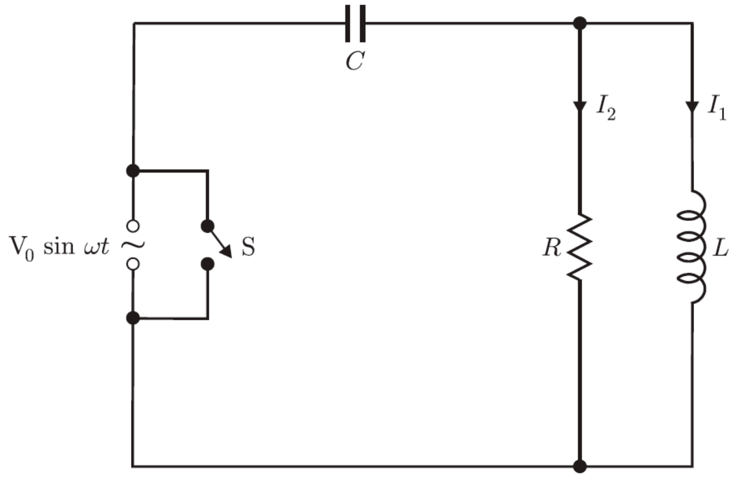
where $0 < M < L$. Find formulae for the two possible frequencies at which the coupled system may oscillate sinusoidally.

10. Solve the differential equations

$$2\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} - 3\frac{\mathrm{d}y}{\mathrm{d}x} + 2\frac{\mathrm{d}z}{\mathrm{d}x} + 3y + z = \mathrm{e}^{2x}$$
$$\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} - 3\frac{\mathrm{d}y}{\mathrm{d}x} + \frac{\mathrm{d}z}{\mathrm{d}x} + 2y - z = 0.$$

Is it possible to have a solution to these equations for which $y = z = 0$ when $x = 0$?

11. An alternating voltage $V = V_0 \sin \omega t$ is applied to the circuit below.



The following equations may be derived from Kirchoff's laws:

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\begin{aligned}
I_2 R + \frac{Q}{C} &= V, \\
L \frac{\mathrm{d}I_1}{\mathrm{d}t} &= I_2 R, \\
\frac{\mathrm{d}Q}{\mathrm{d}t} &= I_1 + I_2,
\end{aligned}
$$

where Q is the charge on the capacitor.

Derive a second-order differential equation for I_1, and hence obtain the steady state solution for I_1 after transients have decayed away.

Determine the angular frequency ω at which I_1 is in phase with V, and obtain expressions for the amplitudes of I_1 and I_2 at this frequency.

Suppose now that the switch S is closed and the voltage supply removed when I_1 is at its maximum value. Obtain the solution for the subsequent variation of I_1 with time for the case $L = 4CR^2$, and sketch the form of your solution.