Theory confronts experiment - Cross sections and decay rates

Scattering in Quantum Mechanics

- Prepare state at $t = -\infty$ $\quad |\psi_{in}(t = -\infty)> = |i>$
- Time evolution (possibly scattering) $\quad |\psi_{in}(t = +\infty)> = S\,|\psi_{in}(t = -\infty)>$
- Observe resulting system in state $\quad |\psi_{out}(t = +\infty)> = |f>$

QM : probability amplitude :

$$<\psi_{out}(t = +\infty)\,|\,\psi_{in}(t = +\infty)> = <\psi_{out}(t = +\infty)\,|\,S\,|\,\psi_{in}(t = -\infty)>$$

$$= <f\,|\,S\,|\,i> = S_{fi}$$

$$S_{fi} = \delta_{fi} + iT_{fi}$$

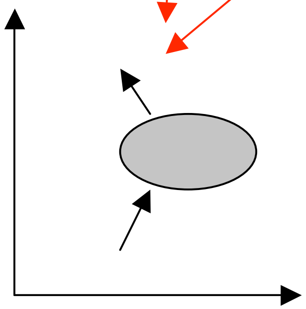
S matrix for Klein Gordon scattering

Relativistic probability density

$$S_{p'+,p+} = <\psi_{out}(t=+\infty)\,|\,\psi_{in}(t=+\infty)> \quad = \lim_{t\to\infty}\int d^3x\; f^{+*}_{p'+}\, i\partial_0 \psi(x)$$

$$\psi(x) = \phi(x) - \int d^4x'\,\Delta_F(x'-x)V(x')\phi(x') + \dots$$

$$\Delta_F(x'-x) = i\int d^3p\; f^{+}_{p}(x')f^{+*}_{p}(x)\theta(t'-t) + i\int d^3p\; f^{-}_{p}(x')f^{-*}_{p}(x)\theta(t-t')$$



Physical interpretation of Quantum Mechanics

Schrödinger equation (S.E.)

$$i\frac{\partial\phi}{\partial t}+\frac{1}{2m}\nabla^2\phi=0$$

$$i\phi^*(S.E.)-i\phi(S.E.)^* \Rightarrow \frac{\partial\rho}{\partial t}+\nabla.\mathbf{j}=0 \text{ continuity eq.}$$

$$\rho=|\phi|^2$$

"probability density"

$$\mathbf{j}=-\frac{i}{2m}(\phi^*\nabla\phi-\phi\nabla\phi^*)$$

"probability current"

Klein Gordon equation

$$-\frac{\partial^2\phi}{\partial t^2}+\nabla^2\phi=m^2\phi$$

$$\rho=i\left(\phi^*\frac{\partial\phi}{\partial t}-\phi\frac{\partial\phi^*}{\partial t}\right)$$

$$\mathbf{j}=-i(\phi^*\nabla\phi-\phi\nabla\phi^*)$$

$$j^\mu=(\rho,\mathbf{j})$$

$$\phi=Ne^{-ip.x},\quad \rho=2E|N|^2$$

$$\int_V\rho\,dV=\int\rho\,d^3x=2E$$

$$f_p^\pm=e^{\mp ip.x}\frac{1}{\sqrt{2p^0V}}$$

Normalised free particle solutions

S matrix for Klein Gordon scattering

Relativistic probability density

$$S_{p'+,p+} = \langle \psi_{out}(t=+\infty) | \psi_{in}(t=+\infty) \rangle \quad = \lim_{t\to\infty} \int d^3x \; f^{+*}_{p'+} i\partial_0 \psi(x)$$

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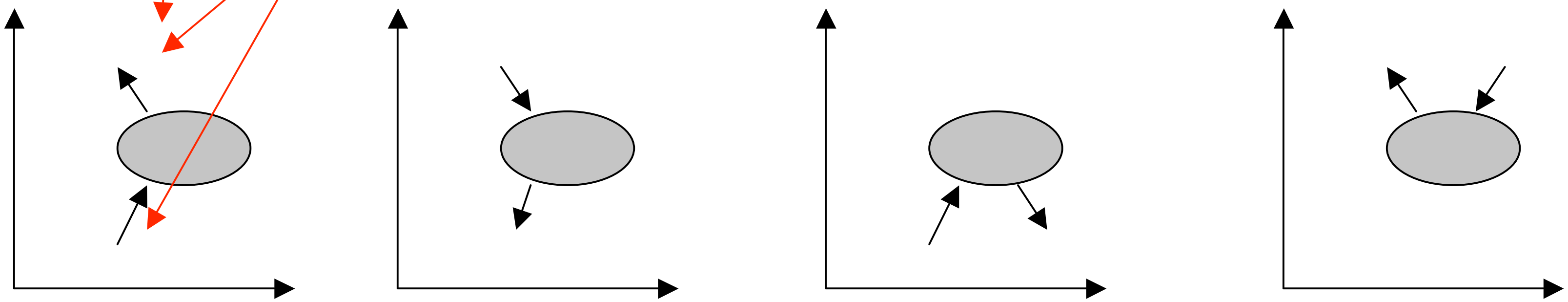
$$S_{p'+,p+} = \delta^3(p'_+ - p_+) - i\int d^4x' \; f^{+*}_{p'+}(x') V(x') \; f^{+}_{p+}(x') + \ldots$$



Feynman rules

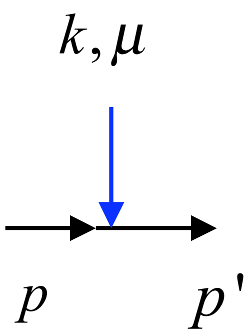
$$iT_{fi} = -i\int d^4y\; f^{+*}_{p'+}(y)\; V(y)\; f^{+}_{p+}(y) = i\int d^4y\; f^{+*}_{p'+}(y)\; ie(A^\mu\partial_\mu + \partial_\mu A^\mu)\; f^{+}_{p+}(y)$$

$$f^{\pm}_{p} = e^{\mp ip.x}\frac{1}{\sqrt{2p^0V}}$$

$$= -i\int d^4y\; j^{fi}_\mu A^\mu$$

$$j^{fi}_\mu = -ie\Big(f^{+*}_{p'+}(y)\big[\partial_\mu f^{+}_{p+}(y)\big] - \big[\partial_\mu f^{*+}_{p'+}(y)\big] f^{+}_{p+}(y)\Big)$$

$$= -e(p_f + p_i)_\mu e^{i(p_f - p_i).y}$$



$$ie(p_\mu + p'_\mu)$$

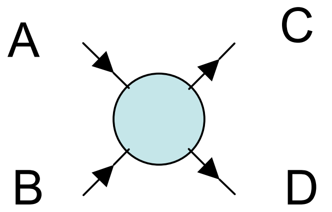
Feynman rule associated with Feynman diagram

The transition rate

$$T_{fi} = -\int d^4x\, \phi_f^*(x) V(x) \phi_i(x) + \dots$$

$$\phi_{i,f} \to f_p^{\pm} = e^{\mp ip.x} \frac{1}{\sqrt{2p^0 V}} \equiv \frac{N}{\sqrt{V}} e^{\mp ip.x}$$

e.g.



Transition rate per unit volume

$$W_{fi} = \frac{|T_{fi}|^2}{TV}$$

$$\phi_{f,i} = e^{\mp ip.x}$$

$$T_{fi} = -\frac{N_A N_B N_C N_D}{V^2} (2\pi)^4 \delta^4(p_C + p_D - p_A - p_B) \mathfrak{M}_{\mathrm{fi}}$$

$$W_{fi} = (2\pi)^4 \frac{\delta^4(p_C + p_D - p_A - p_B) |\mathfrak{M}|^2}{V^4} \left(\frac{1}{2E_A}\right)\left(\frac{1}{2E_B}\right)\left(\frac{1}{2E_C}\right)\left(\frac{1}{2E_D}\right)$$

(c.f. Fermi's golden rule)

Fundamental experimental objects

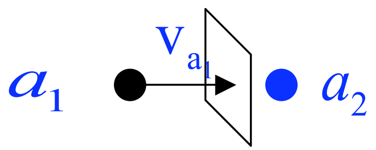
$\Gamma(a_1 \rightarrow b_1 b_2 \ldots b_n)$ **Decay width = 1/lifetime**

$\sigma(a_1 a_2 \rightarrow b_1 b_2 \ldots b_n)$ **Cross section**

$$\prod_{i=1}^{n} \frac{V d^3 p}{(2\pi)^3}$$

Cross section = $\dfrac{\text{Transition rate x Number of final states}}{\text{Initial flux}}$

$$\frac{|v_{a_1}|}{V} \times \frac{1}{V}$$



(Lab frame)

particles passing through unit area in unit time

target particles per unit volume

The cross section

$$\text{Cross section} = \frac{\text{Transition rate} \times \text{Number of final states}}{\text{Initial flux}}$$

$$d\sigma = \frac{V^2}{|\mathbf{v_A}| 2E_A 2E_B} \frac{1}{V^4} |\mathfrak{M}|^2 \frac{(2\pi)^4}{(2\pi)^6} \delta^4(p_C + p_D - p_A - p_B) \frac{d^3 p_C}{2E_C} \frac{d^3 p_D}{2E_D} V^2$$

$$d\sigma = \frac{|\mathfrak{M}|^2}{F} dQ$$

$$dQ = (2\pi)^4 \delta^4(p_C + p_D - p_A - p_B) \frac{d^3 p_C}{(2\pi)^3 2E_C} \frac{d^3 p_D}{(2\pi)^3 2E_D}$$

Lorentz Invariant Phase space

$$F = |\mathbf{v_A}| 2E_A 2E_B$$
$$= 4((p_A . p_B)^2 - m_A^2 m_B^2)^{1/2}$$

See Halzen & Martin, "Quarks and leptons" Wiley, P90