Relativistic theory of scattering

Scalar particle – satisfies KG equation

$$(\partial_\mu \partial^\mu + m^2)\phi = 0$$

$$\partial^\mu = (\tfrac{1}{c}\tfrac{\partial}{\partial t}, -\nabla), \quad \partial_\mu = (\tfrac{1}{c}\tfrac{\partial}{\partial t}, \nabla)$$

- Classical electrodynamics, motion of charge –e in EM potential $\mathrm{A}^\mu = (\mathrm{A}^0, \mathbf{A})$

is obtained by the substitution : $p^\mu \to p^\mu + eA^\mu$

- Quantum mechanics : $i\partial^\mu \to i\partial^\mu + eA^\mu$

The Klein Gordon equation becomes:

$$(\partial_\mu \partial^\mu + m^2)\psi = -V\psi \quad where \quad V = -ie(\partial_\mu A^\mu + A^\mu \partial_\mu) - e^2 A^2$$

The smallness of the EM coupling, $\alpha_{em} = \frac{e^2}{4\pi} \sim \frac{1}{137}$, means that it is sensible to

Make a “perturbation” expansion of V in powers of α_{em}

Want to solve : $(\partial_\mu \partial^\mu + m^2)\psi = -V\psi$

Solution : $\psi(x) = \phi(x) - \int d^4x' \Delta_F(x'-x) V(x')\psi(x')$

where $(\partial_\mu \partial^\mu + m^2)\phi = 0$

and $(\partial_\mu \partial^\mu + m^2)\Delta_F(x'-x) = \delta^4(x'-x)$

Feynman propagator

Dirac Delta function

In bra- ket- notation

$$|\psi> = |\phi> - \frac{1}{\partial^2 + m^2 + i\varepsilon} V |\psi>$$

$$\psi(x) = \phi(x) - \int d^4x' < x | \frac{1}{\partial^2 + m^2 + i\varepsilon} | x' >< x' | V | \psi >$$

The propagator of the Klein Gordon particle

$$\Delta_F(x'-x) = \langle x | \frac{1}{\partial^2 + m^2 + i\varepsilon} | x' \rangle$$

$$= \int d^4p' \int d^4p'' \langle x | p' \rangle \langle p' | \frac{1}{\partial^2 + m^2 + i\varepsilon} | p'' \rangle \langle p'' | x' \rangle$$

$$\underbrace{\langle p' | \frac{1}{\partial^2 + m^2 + i\varepsilon} | p'' \rangle} = \frac{\delta^4(p'-p'')}{-p'^2 + m^2 + i\varepsilon}$$

$$= \int d^4p' \left(\frac{1}{2\pi}\right)^2 e^{ip'.x} \frac{1}{-p'^2 + m^2 + i\varepsilon} \left(\frac{1}{2\pi}\right)^2 e^{-ip'.x'}$$

$$\boxed{\Delta_F(x) = -\frac{1}{(2\pi)^4} \int d^4p' \, e^{-ip'.x} \frac{1}{p'^2 - m^2 - i\varepsilon}}$$

Want to solve :

$$(\partial_\mu \partial^\mu + m^2)\psi = -V\psi$$

Solution :

$$\psi(x) = \phi(x) - \int d^4x' \Delta_F(x'-x) V(x')\psi(x')$$

where

$$(\partial_\mu \partial^\mu + m^2)\phi = 0$$

and

$$(\partial_\mu \partial^\mu + m^2)\Delta_F(x'-x) = \delta^4(x'-x)$$

Equivalent method : solve for propagator in momentum space by taking Fourier transform

$$\frac{1}{(2\pi)^2}\int e^{-ip.(x'-x)} \quad (\partial_\mu \partial^\mu + m^2)\Delta_F(x'-x) d^4(x'-x) = \frac{1}{(2\pi)^2}\int e^{-ip.(x'-x)}\delta^4(x'-x) d^4(x'-x)$$

$$\Rightarrow \left(-p^2 + m^2\right)\Delta_F(p) = \frac{1}{\left(2\pi^2\right)}$$

$$\tilde{\Delta}_F(p) = \frac{1}{(2\pi)^2}\frac{1}{-p^2 + m^2 + i\varepsilon}, \qquad \Delta_F(x) = -\frac{1}{(2\pi)^4}\int d^4p\, e^{-ip.x}\frac{1}{p^2 - m^2 - i\varepsilon}$$

$$\Delta_F(x) = -\frac{1}{(2\pi)^4}\int d^4p\, e^{-ip.x}\, \frac{1}{p^2 - m^2 - i\varepsilon}$$

$$p^2 + m^2 - i\varepsilon \quad \Rightarrow \quad p_o^2 = \underline{p}^2 + m^2 - i\varepsilon \quad \Rightarrow \quad p_0 = \pm\left(\underline{p}^2 + m^2\right)^{1/2} \mp i\delta = \pm\omega_p \mp i\delta$$

$$\Delta_F(x'-x) = -\frac{1}{(2\pi)^4}\int d^3p\, e^{-i\underline{p}.(\underline{x}'-\underline{x})}\int dp_0 \frac{e^{-ip_0(t'-t)}}{\left(p_0 - (\omega_p - i\delta)\right)\left(p_0 - (-\omega_p + i\delta)\right)}$$

I

- If $t' - t > 0$, choose contour such that $p_0 = -ip_I \quad (p_I \text{ +ve}) \Rightarrow e^{-ip_0(t'-t)} = e^{-p_I(t'-t)}$

$$I = -\frac{\pi i}{\omega_p} e^{-i\omega_p(t'-t)}\theta(t'-t)$$

- If $t' - t < 0$, choose contour such that $p_0 = +ip_I \quad (p_I \text{ +ve}) \Rightarrow e^{+ip_0(t'-t)} = e^{-p_I(t'-t)}$

$$I = -\frac{\pi i}{\omega_p} e^{+i\omega_p(t'-t)}\theta(t-t')$$

Feynman propagator

$$\Delta_F(x-x') = \frac{i}{(2\pi)^3\, 2\omega_p}\int d^3p \left[e^{-i\omega_p(t'-t)+i\underline{p}.(\underline{x}'-\underline{x})}\theta(t'-t) + e^{+i\omega_p(t'-t)+i\underline{p}.(\underline{x}'-\underline{x})}\theta(t-t')\right]$$

An alternative form :

$$f_p^{\pm} = e^{\mp ip.x}\frac{1}{\sqrt{2p^0V}} = e^{\mp ip.x}\frac{1}{\sqrt{2p^0(2\pi)^3}}$$

$$\Delta_F(x-x') = i\int d^3p\, f_p^{+}(x') f_p^{+*}(x)\theta(t'-t) + i\int d^3p\, f_p^{-}(x') f_p^{-*}(x)\theta(t-t')$$

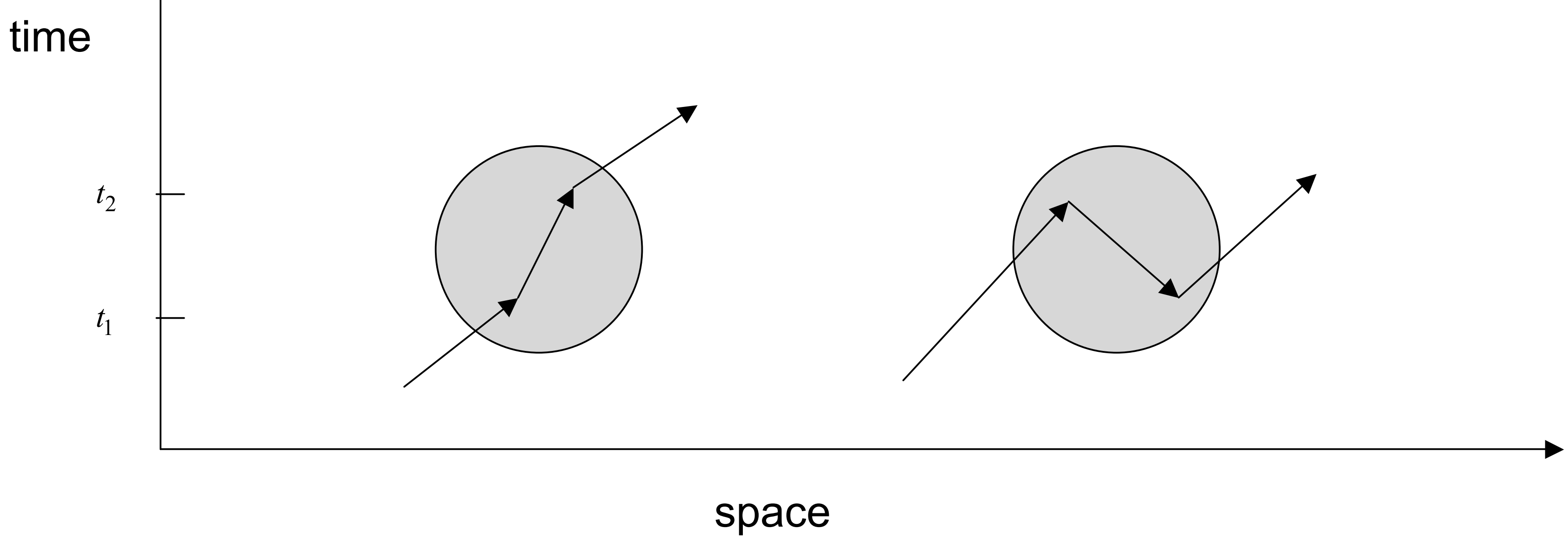
Feynman – Stuckelberg interpretation

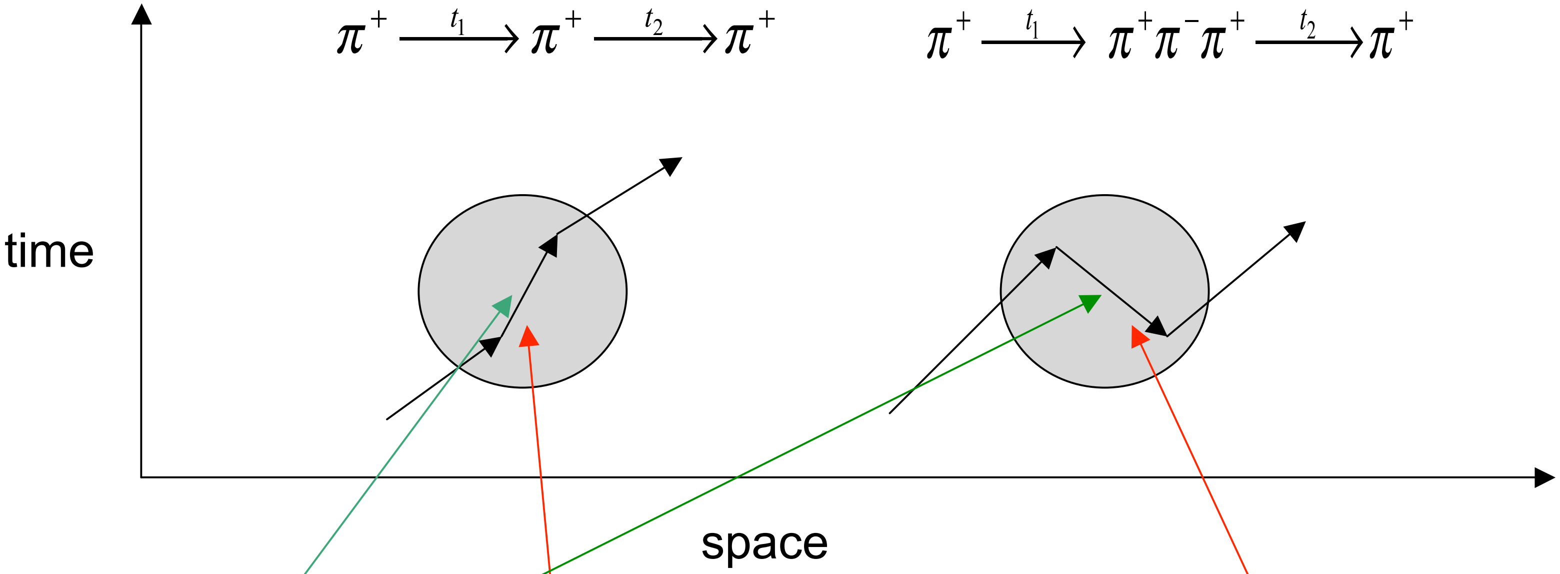
$$\pi^+(E>0)\ \uparrow \quad \equiv \quad \pi^-(E<0)\ \downarrow$$

$$e^{-iEt} \qquad\qquad e^{-i(-E)(-t)}$$

Two different time orderings giving same observable event :

$$\pi^+ \xrightarrow{t_1} \pi^+ \xrightarrow{t_2} \pi^+ \qquad\qquad \pi^+ \xrightarrow{t_1} \pi^+\pi^-\pi^+ \xrightarrow{t_2} \pi^+$$





$$\Delta_F(x'-x) = \frac{1}{(2\pi)^4}\int d^4p\; e^{-ip.(x'-x)}\,\frac{1}{p^2-m^2-i\varepsilon} = i\int \frac{d^3p}{(2\pi)^3\,2p^0}\, e^{-ip^0|t'-t|-ip.(x'-x)}$$

$$\Delta_F(x-x') = i\int d^3p\; f_p^{+}(x')\,f_p^{+*}(x)\,\theta(t_2-t_1) + i\int d^3p\; f_p^{-}(x')\,f_p^{-*}(x)\,\theta(t_1-t_2)$$

where $f_p^{\pm} = e^{\mp ip.x}\frac{1}{\sqrt{2p^0V}}$ are positive and negative energy solutions to free KG equation