Relativistic quantum mechanics

Quantum Mechanics

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Relativity

} Quantum Field theory

Special relativity

- Space time point $a^{\mu} = (ct, x, y, z)$ not invariant under translations

- Space-time vector $(a+\Delta a)^{\mu} - a^{\mu} = \Delta a^{\mu} = (c\Delta t, \Delta x, \Delta y, \Delta z)$

  Invariant under translations …but not invariant under rotations or boosts

- Einstein postulate : the real invariant distance is

$$\left(\Delta a^{0}\right)^{2} - \left(\Delta a^{1}\right)^{2} - \left(\Delta a^{2}\right)^{2} - \left(\Delta a^{3}\right)^{2} = \sum_{\mu,\nu=0}^{3} g_{\mu\nu} \Delta a^{\mu} \Delta a^{\nu} = \Delta a^{\mu} \Delta a_{\mu} = (\Delta a)^{2}$$

$$g_{\mu\nu} = diag(+1,-1,-1,-1) = g^{\mu\nu}$$

- Physics invariant under all transformations that leave all such distances invariant :

  Translations and Lorentz transformations

Lorentz transformations :

$$x^{\mu} \rightarrow \Lambda^{\mu}_{\nu} x^{\nu} = \sum_{\nu=0}^{3} \Lambda^{\mu}_{\nu} x^{\nu} = \tilde{x}^{\mu} \quad \Rightarrow \quad g_{\mu\nu} \tilde{x}^{\mu} \tilde{x}^{\nu} = g_{\mu\nu} x^{\mu} x^{\nu} \quad \Rightarrow \quad g_{\mu\nu} \Lambda^{\mu}_{\alpha} \Lambda^{\nu}_{\beta} = g_{\alpha\beta}$$

(Summation assumed)

Solutions :

- 3 rotations R

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta & 0 & \sin\theta \\ 0 & 0 & 1 & 0 \\ 0 & -\sin\theta & 0 & \cos\theta \end{pmatrix}$$

- 3 boosts B

$$\begin{pmatrix} \cosh\alpha & \sinh\alpha & 0 & 0 \\ \sinh\alpha & \cosh\alpha & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

- Space reflection – parity P

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

- Time reflection, time reversal T

$$\begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

4 vector notation

$$A^\mu = (A^0, \underline{A}), \quad B^\mu = (B^0, \underline{B})$$ contravariant

$$A_\mu = (A^0, \underline{-A}), \quad B_\mu = (B^0, -\underline{B})$$ covariant

$$A_\mu = g_{\mu\nu} A^\nu \qquad A^\mu = g^{\mu\nu} A_\nu$$

$$A.B = A_\mu B^\mu = A^\mu B_\mu = A^0 B^0 - \underline{A}.\underline{B}$$

4 vectors

$$(ct, \underline{x}) \equiv x^\mu \qquad (\frac{E}{c}, \underline{p}) \equiv p^\mu$$

$$\partial^\mu = (\frac{\partial}{c\partial t}, -\underline{\nabla}) \qquad \partial_\mu = (\frac{\partial}{c\partial t}, \underline{\nabla})$$

$$p^\mu \to i\hbar\partial^\mu \quad -p_\mu p^\mu = E^2 - \underline{p}^2 \quad \to \quad -\Box^2 \equiv \partial_\mu \partial^\mu$$

The Klein Gordon equation (1926)

Scalar field (J=0) : $\phi(\mathbf{x})$

$$E^2 = \mathbf{p}^2 + m^2 \quad \Rightarrow \quad -\frac{\partial^2 \phi}{\partial t^2} + \nabla^2 \phi = m^2 \phi$$ (Natural units)

$$-\partial_\mu \partial^\mu \phi = -\Box^2 \phi = m^2 \phi$$

Energy eigenvalues $E = \pm(\mathbf{p}^2 + m^2)^{1/2}$???

Physical interpretation of Quantum Mechanics

Schrödinger equation (S.E.)

$$i\frac{\partial\phi}{\partial t} + \frac{1}{2m}\nabla^2\phi = 0$$

$$i\phi^*(S.E.) - i\phi(S.E.)^* \Rightarrow \frac{\partial\rho}{\partial t} + \nabla.\mathbf{j} = 0 \text{ continuity eq.}$$

$$\rho = |\phi|^2$$

"probability density"

$$\mathbf{j} = -\frac{i}{2m}(\phi^*\nabla\phi - \phi\nabla\phi^*)$$

"probability current"

Klein Gordon equation

$$-\frac{\partial^2\phi}{\partial t^2} + \nabla^2\phi = m^2\phi$$

$$\rho = i\left(\phi^*\frac{\partial\phi}{\partial t} - \phi\frac{\partial\phi^*}{\partial t}\right)$$

$$\mathbf{j} = -i(\phi^*\nabla\phi - \phi\nabla\phi^*)$$

$$j^\mu = (\rho, \mathbf{j})$$

$$\phi = Ne^{-ip.x}, \quad \rho = 2E|N|^2$$

$$\int_V \rho\, dV = \int \rho\, d^3x = 2E$$

$$f_p^{\pm} = e^{\mp ip.x}\frac{1}{\sqrt{2p^0 V}}$$

Normalised free particle solutions

Physical interpretation of Quantum Mechanics

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Klein Gordon equation

$$-\frac{\partial^2\phi}{\partial t^2}+\nabla^2\phi=m^2\phi$$

$$\rho=i\left(\phi^*\frac{\partial\phi}{\partial t}-\phi\frac{\partial\phi^*}{\partial t}\right)$$

$$\mathbf{j}=-i(\phi^*\nabla\phi-\phi\nabla\phi^*)$$

$$j^\mu=(\rho,\mathbf{j})$$

$$\phi=Ne^{-ip.x},\quad \rho=2E|N|^2$$

Negative probability?

$$f_p^\pm=e^{\mp ip.x}\frac{1}{\sqrt{2p^0V}}$$

Normalised free particle solutions

Relativistic QM - The Klein Gordon equation (1926)

Scalar particle (field) (J=0) : $\phi(\mathbf{x})$

$$E^2 = \mathbf{p}^2 + m^2 \quad \Rightarrow \quad -\frac{\partial^2\phi}{\partial t^2} + \nabla^2\phi = m^2\phi$$

Energy eigenvalues $\underline{E = \pm(\mathbf{p}^2 + m^2)^{1/2}}$???

1927 Dirac tried to eliminate negative solutions by writing a relativistic equation linear in E (a theory of fermions)

1934 Pauli and Weisskopf revived KG equation with E<0 solutions as E>0 solutions for particles of opposite charge (antiparticles). Unlike Dirac's hole theory this interpretation is applicable to bosons (integer spin) as well as to fermions (half integer spin).

As we shall see the antiparticle states make the field theory causal

But energy eigenvalues $E = \pm(\mathbf{p}^2 + m^2)^{1/2}$???

⇨ Feynman – Stuckelberg interpretation

$$\pi^+(E>0) \uparrow \quad \equiv \quad \pi^-(E<0) \downarrow$$

$$e^{-iEt} \qquad\qquad e^{-i(-E)(-t)}$$

Two different time orderings giving same observable event :

$$\pi^+ \xrightarrow{t_1} \pi^+ \xrightarrow{t_2} \pi^+ \qquad\qquad \pi^+ \xrightarrow{t_1} \pi^+\pi^-\pi^+ \xrightarrow{t_2} \pi^+$$