The Lipmann Schwinger Equation – operator formalism

$$H_0 \,|\, \phi >= E \,|\, \phi > \quad \text{stationary state}$$

$$H_0 + V \,|\, \psi >= E \,|\, \psi > \quad (2.1)$$

Bra- ket- notation … independent of representation choice

$$< x \,|\, \psi >= \psi(x), \qquad < p \,|\, \psi >= \psi(p)$$

Plane wave states : stationary states

$$\underline{P}\,|\,\underline{p}>=\underline{p}\,|\,\underline{p}>$$

$$\underline{X}\,|\,\underline{x}>=\underline{x}\,|\,\underline{x}>$$

$$H_0\,|\,\underline{p}>=\frac{p^2}{2\mu}\,|\,\underline{p}>$$

$$H_0\,|\,\underline{x}>=?$$

Orthonormality and completeness

$$<\underline{p}\,|\,\underline{p}'>=\delta^3(\underline{p}-\underline{p}')$$

$$<\underline{x}\,|\,\underline{x}'>=\delta^3(\underline{x}-\underline{x}')$$

$$\int d^3p\,|\,\underline{p}><\underline{p}\,|=1$$

$$\int d^3x\,|\,\underline{x}><\underline{x}\,|=1$$

$$<\underline{x}\,|\,\underline{p}>=\left(\frac{1}{2\pi\hbar}\right)^{3/2}e^{i\underline{x}.\underline{p}/\hbar}$$

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$$H_0 \,|\, \phi >= E \,|\, \phi > \quad \text{stationary state}$$

$$H_0 + V \,|\, \psi >= E \,|\, \psi > \quad (2.1)$$

Solution to (2.1)

$$|\, \psi >=|\, \phi > + \frac{1}{E - H_0} V \,|\, \psi > \qquad (2.2)$$

(up to complications coming from singular nature of the inverse operator $\frac{1}{E - H_0}$)

We will prove a consistent regularisation procedure is

$$|\, \psi_{\pm} >=|\, \phi > + \frac{1}{E - H_0 \pm i\varepsilon} V \,|\, \psi_{\pm} > \qquad (2.3)$$

$$|\psi_{\pm}\rangle = |\phi\rangle + \frac{1}{E - H_0 \pm i\varepsilon} V |\psi_{\pm}\rangle \qquad (2.3)$$

In position basis

$$\langle \underline{x} | \psi_{\pm}\rangle = \langle \underline{x} | \phi\rangle + \langle \underline{x} | \frac{1}{E - H_0 \pm i\varepsilon} V |\psi_{\pm}\rangle$$

$$\psi_{\pm}(\underline{x}) = \phi(\underline{x}) + \int d^3x' \langle \underline{x} | \frac{1}{E - H_0 \pm i\varepsilon} | \underline{x}' \rangle \langle \underline{x}' | V | \psi_{\pm}\rangle$$

$$\langle \underline{x}' | V | \psi_{\pm}\rangle = \int d^3x'' \langle \underline{x}' | V | \underline{x}''\rangle \langle \underline{x}'' | \psi_{\pm}\rangle$$

$$= \int d^3x'' V(\underline{x}')\delta^3(\underline{x}' - \underline{x}'')\psi_{\pm}(\underline{x}'')$$

$$= V(\underline{x}')\psi_{\pm}(\underline{x}')$$

$\langle \underline{x}' | V | \underline{x}''\rangle = V(\underline{x}')\delta^3(\underline{x}' - \underline{x}'')$
if V local

$$\psi_{\pm}(\underline{x}) = \phi(\underline{x}) + \int d^3x' \langle \underline{x} | \frac{1}{E - H_0 \pm i\varepsilon} | \underline{x}' \rangle V(x')\psi_{\pm}(x)$$

The Green function

$$\psi_{\pm}(\underline{x}) \ = \phi(\underline{x}) + \int d^3x' < \underline{x} \,|\, \frac{1}{E - H_0 \pm i\varepsilon} \,|\, \underline{x}' > V(x')\psi_{\pm}(\underline{x}')$$

$$c.f. \qquad \mathrm{v}_{\mathrm{k}}^{\mathrm{diffractive}}(\underline{r}) = e^{ikz} + \int d^3r' \frac{2\mu}{\hbar^2} G_{+}(\underline{r} - \underline{r}')\, V(\underline{r}')\, \mathrm{v}_{\mathrm{k}}^{\mathrm{diffractive}}(\underline{r}') \qquad (1.14)$$

$$\Rightarrow \ G_{\pm}(\underline{x}, \underline{x}') = \frac{\hbar^2}{2\mu} < \underline{x} \,|\, \frac{1}{E - H_0 \pm i\varepsilon} \,|\, \underline{x}' > \qquad (2.5)$$

$$? = -\frac{1}{4\pi} \frac{e^{\pm ik|\underline{x} - \underline{x}'|}}{|\underline{x} - \underline{x}'|} \qquad (1.15) \qquad \left(E \equiv \frac{\hbar^2 k^2}{2\mu} \right)$$

Determination of the Green function

$$G_\pm(\underline{x},\underline{x}') = \frac{\hbar^2}{2\mu} < \underline{x} | \frac{1}{E - H_0 \pm i\varepsilon} | \underline{x}' > \qquad (2.5)$$

$$= \frac{\hbar^2}{2\mu} \int d^3p \int d^3p' < \underline{x} | \underline{p} >< \underline{p} | \frac{1}{E - H_0 \pm i\varepsilon} | \underline{p}' >< \underline{p}' | \underline{x}' >$$

$$< \underline{p} | \frac{1}{E - H_0 \pm i\varepsilon} | \underline{p}' > = \frac{\delta^3(\underline{p} - \underline{p}')}{E - \frac{p'^2}{2\mu} \pm i\varepsilon}$$

$$= \int d^3p \left(\frac{1}{2\pi\hbar}\right)^{3/2} e^{i\underline{x}.\underline{q}} \frac{1}{k^2 - q^2 \pm i\varepsilon'} \left(\frac{1}{2\pi\hbar}\right)^{3/2} e^{-i\underline{x}'.\underline{q}}$$

$$E \equiv \frac{\hbar^2 k^2}{2\mu}, \quad \underline{p} = \hbar \underline{q}$$

$$= \int d^3q \left(\frac{1}{2\pi}\right)^3 e^{i(\underline{x}-\underline{x}').\underline{q}} \frac{1}{k^2 - q^2 \pm i\varepsilon'}$$

$$G_\pm(\underline{x},\underline{x}') \quad = \int d^3q \left(\frac{1}{2\pi}\right)^3 e^{i(\underline{x}-\underline{x}').\underline{q}} \frac{1}{k^2 - q^2 \pm i\varepsilon'}$$

$$= \frac{1}{(2\pi)^3} \int_0^\infty q^2 dq \int_0^{2\pi} d\phi \int_{-1}^{1} d(\cos\theta) \frac{e^{iq|\vec{x}-\vec{x}'|\cos\theta}}{k^2 - q^2 \pm i\varepsilon'}$$

$$= -\frac{(2/2)}{8\pi^2} \frac{1}{i|\vec{x}-\vec{x}'|} \int_{-\infty}^{\infty} \frac{q\,dq(e^{iq|\vec{x}-\vec{x}'|} - e^{-iq|\vec{x}-\vec{x}'|})}{q^2 - k^2 \mp i\varepsilon'}$$

Simple poles at $q_\pm = \pm k\sqrt{1 \pm \left(\frac{i\varepsilon'}{k^2}\right)} \approx \pm(k \pm i\varepsilon'')$ $\qquad \varepsilon'' k = \varepsilon'/2$

$$q^2 - k^2 \mp i\varepsilon' = (q - q_+)(q - q_-)$$

Determination of the Green function

$$G_{+}(\underline{x},\underline{x}') = \frac{\hbar^2}{2\mu} < \underline{x} \,|\, \frac{1}{E - H_0 \pm i\varepsilon} \,|\, \underline{x}' > \qquad (2.5)$$

$$= -\frac{1}{8\pi^2}\frac{1}{i\,|\vec{x}-\vec{x}'|}\int_{-\infty}^{\infty} \frac{q\,dq\,(e^{iq|\vec{x}-\vec{x}'|} - e^{-iq|\vec{x}-\vec{x}'|})}{q^2 - k^2 \mp i\varepsilon'}$$

$$= -\frac{1}{4\pi}\frac{e^{\pm ik|\underline{x}-\underline{x}'|}}{|\underline{x}-\underline{x}'|} \qquad (1.15)$$

$$\psi_{\pm}(\underline{x}) = \phi(\underline{x}) - \frac{1}{4\pi}\frac{2\mu}{\hbar^2}\int d^3x' \frac{e^{\pm ik|\underline{x}-\underline{x}'|}}{|\underline{x}-\underline{x}'|} V(x')\psi_{\pm}(\underline{x}')$$

c.f.eq(1.14)

The Born Series in operator notation

$$|\psi_{\pm}>=|\phi>+\frac{1}{E-H_0+i\varepsilon}V|\psi_{+}> \qquad (2.3)$$

Define the transition operator T by

$$V|\psi_{+}>=T|\phi> \qquad (2.8)$$

$$\Rightarrow \quad V|\psi_{+}>=T|\phi>=V|\phi>+V\frac{1}{E-H_0\pm i\varepsilon}T|\phi>$$

$$T=V+V\frac{1}{E-H_0+i\varepsilon}T \qquad (2.9)$$

Since $|\phi>$ is a complete set of states

Born series :

$$\mathrm{T}=V+V\frac{1}{E-H_0+i\varepsilon}V+\ V\frac{1}{E-H_0+i\varepsilon}V\frac{1}{E-H_0+i\varepsilon}V+\ \ldots \qquad (2.10)$$

The scattering amplitude

$$f_k(\theta,\phi) \equiv f(\underline{k}_i,\underline{k}_d) = -\frac{1}{4\pi}\frac{2\mu}{\hbar^2}\int d^3x' e^{-i\underline{k}_d\cdot\underline{x}'}\, V(\underline{x}')\, \mathrm{v}_{\mathrm{k}}^{\mathrm{diffractive}}(\underline{x}') \qquad (1.16)$$

$$= -\frac{1}{4\pi}\frac{2\mu}{\hbar^2}(2\pi)^3\int d^3x' \frac{e^{-i\underline{k}_d\cdot\underline{x}'}}{(2\pi)^{3/2}}\, V(\underline{x}')\, \frac{\psi(\underline{x}')}{(2\pi)^{3/2}}$$

$$<\underline{k}_d\,|\,\underline{x}> \quad <\underline{x}\,|\,V\,|\,\underline{x}'> \quad <\underline{x}'\,|\,\psi>$$

$$= -\frac{1}{4\pi}\frac{2\mu}{\hbar^2}(2\pi)^3 <\underline{k}'\,|\,V\,|\,\psi_+> = -\frac{1}{4\pi}\frac{2\mu}{\hbar^2}(2\pi)^3 <\underline{k}'\,|\,T\,|\,\underline{k}>$$

$$(V\,|\,\psi_+> = T\,|\,\phi> \qquad (2.8))$$

$$f(\underline{k}_i, \underline{k}_d) = -\frac{1}{4\pi}\frac{2\mu}{\hbar^2}(2\pi)^3 < \underline{k}' | T | \underline{k} >$$

The Born series

$$\mathrm{T} = V + V \frac{1}{E - H_0 + i\varepsilon} V + V \frac{1}{E - H_0 + i\varepsilon} V \frac{1}{E - H_0 + i\varepsilon} V + \ldots \qquad (2.10)$$

The Born series for the scattering amplitude

$$f(\underline{k}_i, \underline{k}_d) = \sum_{n=1}^{\infty} f^{(n)}(\underline{k}_i, \underline{k}_d)$$

$$f^{(1)}(\underline{k}_i, \underline{k}_d) = -\frac{1}{4\pi}\frac{2\mu}{\hbar^2}(2\pi)^3 < \underline{k}' | V | \underline{k} >$$

$$f^{(1)}(\underline{k}_i, \underline{k}_d) = -\frac{1}{4\pi}\frac{2\mu}{\hbar^2}(2\pi)^3 < \underline{k}' | V \frac{1}{E - H_0 + i\varepsilon} V | \underline{k} > \qquad (2.11)$$