The Born Approximation

$$\mathrm{v}_{\mathrm{k}}^{\mathrm{diffractive}}(\underline{r}) = e^{i\underline{k}_i.\underline{r}} + \int d^3r'\, G_+(\underline{r}-\underline{r}')\, U(\underline{r}')\, \mathrm{v}_{\mathrm{k}}^{\mathrm{diffractive}}(\underline{r}') \qquad (1.14)$$

$$\mathrm{v}_{\mathrm{k}}^{\mathrm{diffractive}}(\underline{r}') = e^{i\underline{k}_i.\underline{r}'} + \int d^3r''\, G_+(\underline{r}'-\underline{r}'')\, U(\underline{r}'')\, \mathrm{v}_{\mathrm{k}}^{\mathrm{diffractive}}(\underline{r}'')$$

Born Expansion

Born approximation

$$\mathrm{v}_{\mathrm{k}}^{\mathrm{diffractive}}(\underline{r}) = e^{i\underline{k}_i.\underline{r}} + \overbrace{\int d^3r'\, G_+(\underline{r}-\underline{r}')\, U(\underline{r}') e^{i\underline{k}_i.\underline{r}'}}$$

$$+ \int d^3r' \int d^3r''\, G_+(\underline{r}-\underline{r}')\, U(\underline{r}') G_+(\underline{r}'-\underline{r}'') U(\underline{r}'') \mathrm{v}_{\mathrm{k}}^{\mathrm{diffractive}}(\underline{r}'') \qquad (1.18)$$

Repeat insertion

$$f(\theta,\phi) = -\frac{1}{4\pi}\int d^3r'\, e^{-ik\hat{\underline{r}}.\underline{r}'}\, U(\underline{r}')\, \mathrm{v}_{\mathrm{k}}^{\mathrm{diffractive}}(\underline{r}')$$

$$f^{Born}(\theta,\phi) = -\frac{1}{4\pi}\int d^3r'\, e^{-ik\hat{\underline{r}}.\underline{r}'}\, U(\underline{r}')\, e^{i\underline{k}_i.\underline{r}'} \qquad (1.19)$$

The Born series

$$\psi(x) = \phi(x) - \int d^3x' G(x'-x) V(x') \psi(x')$$

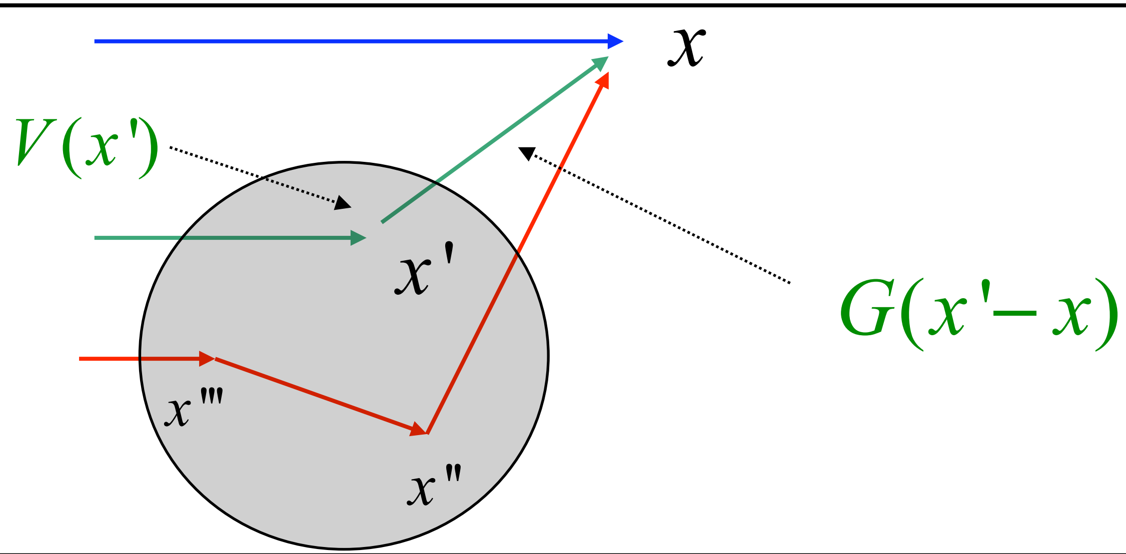
Since V(x) is small can solve this equation iteratively :

$$\psi(x) = \phi(x) - \int d^4x' G(x'-x) V(x') \phi(x')$$

$$+ \int d^4x'' \int d^4x''' G(x''-x) V(x'') G(x'''-x'') V(x''') \phi(x''')$$

$$+ \ldots.$$

Interpretation :



$$f^{Born}(\theta,\phi) = -\frac{1}{4\pi}\int d^3r' e^{-ik\hat{\underline{r}}.\underline{r}'} U(\underline{r}')\, e^{i\underline{k}_i.\underline{r}}$$

$$= -\frac{1}{4\pi}\int d^3r' e^{-i\underline{K}.\underline{r}'} U(\underline{r}') \qquad (1.19)$$

$\underline{K} = \underline{k}_d - \underline{k}_i$

$V(\underline{r}) = \frac{\hbar^2}{2\mu}U(\underline{r})$

$$\sigma^{Born}(\theta,\phi) = |\mathrm{f_k}(\theta,\phi)|^2 \qquad (1.9)$$

$$= \frac{\mu^2}{4\pi^2\hbar^4}\left|\int d^3r' e^{-i\underline{K}.\underline{r}'} V(\underline{r}')\right|^2 \qquad (1.20)$$

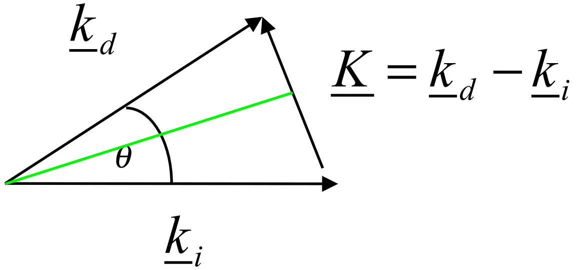
Example : Born approximation for a Yukawa potential

$$V(r) = V_0 \frac{e^{-\alpha r}}{r}$$

$$f^{Born}(\theta,\phi) = -\frac{1}{4\pi}\int d^3r' e^{-i\underline{K}.\underline{r}'}\, U(\underline{r}') \qquad (1.19)$$

$$= -\frac{1}{4\pi}\frac{2\mu V_0}{\hbar^2}\frac{4\pi}{|\underline{K}|}\int_0^\infty r'dr'\sin\left(|\underline{K}||\underline{r}'|\right)\frac{e^{-\alpha r'}}{r'}$$

$$|\underline{k}_d| = |\underline{k}_i|$$

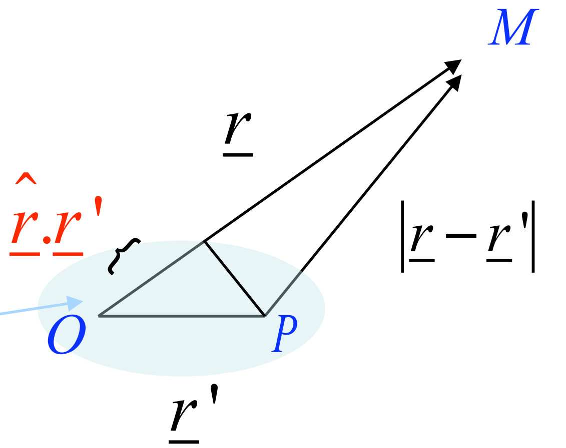


$$f^{Born}(\theta,\phi) = -\frac{2\mu V_0}{\hbar^2}\frac{1}{\alpha^2 + |\underline{K}|^2}$$

$$\sigma(\theta,\phi) = |\mathrm{f}_\mathrm{k}(\theta,\phi)|^2 = \frac{4\mu^2 V_0^{\,2}}{\hbar^4}\frac{1}{\left(\alpha^2 + 4k^2\sin^2\frac{\theta}{2}\right)^2} \qquad (1.9)$$

Validity of Born approximation

$$V(r) = V_0 \frac{e^{-\alpha r}}{r}$$



If BA applicable $\text{v}_\text{k}^{\text{diffractive}}(\underline{r}) \simeq e^{i\underline{k}_i.\underline{r}}$ in

c.f.1.19

$$V(\underline{r}) = \frac{\hbar^2}{2\mu} U(\underline{r})$$

$$G_\pm(\underline{r}-\underline{r}') = -\frac{1}{4\pi} \frac{e^{\pm ik|\underline{r}-\underline{r}'|}}{|\underline{r}-\underline{r}'|}$$

$$\text{v}_\text{k}^{\text{diffractive}}(\underline{r}) = e^{i\underline{k}_i.\underline{r}} + \int d^3r' G_+(\underline{r}-\underline{r}')\, U(\underline{r}') e^{i\underline{k}_i.\underline{r}'}$$

$$+ \int d^3r' \int d^3r'' G_+(\underline{r}-\underline{r}')\, U(\underline{r}') G_+(\underline{r}'-\underline{r}'') U(\underline{r}'')\, \text{v}_\text{k}^{\text{diffractive}}(\underline{r}'') \qquad (1.18)$$

$$|\text{Correction}| = \frac{1}{4\pi}\frac{2\mu}{\hbar^2}\left|\int_0^\infty d^3r' \frac{e^{ikr'}}{r'} V(r') e^{i\underline{k}_i.\underline{r}'}\right| \ll 1 \qquad \text{(evaluated at r=0)}$$

- For $k \ll \alpha$, $\quad e^{i\underline{k}_i.\underline{r}'} \approx 1 \approx e^{ikr'} \quad \Rightarrow \quad \dfrac{2\mu V_0}{\hbar^2 \alpha} \ll 1$ condition for validity of BA
- For large k $\quad \dfrac{2\mu V_0}{\hbar^2 k} \ln\left(\dfrac{k}{\mu}\right) \ll 1$

Sakurai p386

The Lipmann Schwinger Equation – operator formalism

$$H_0 \,|\, \phi >= E \,|\, \phi > \quad \text{stationary state}$$

$$(H_0 + V) \,|\, \psi >= E \,|\, \psi > \quad (2.1)$$

Bra- ket- notation … independent of representation choice

$$< x \,|\, \psi >= \psi(x), \qquad < p \,|\, \psi >= \psi(p)$$

Plane wave states : stationary states

$$\underline{P}\,|\,\underline{p}\,>=\underline{p}\,|\,\underline{p}\,>$$

$$\underline{X}\,|\,\underline{x}\,>=\underline{x}\,|\,\underline{x}\,>$$

$$H_0\,|\,\underline{p}\,>=\frac{p^2}{2\mu}\,|\,\underline{p}\,>$$

$$H_0\,|\,\underline{x}\,>=?$$

Orthonormality and completeness

$$<\underline{p}\,|\,\underline{p}\,'>=\delta^3(\underline{p}-\underline{p}\,')$$

$$<\underline{x}\,|\,\underline{x}\,'>=\delta^3(\underline{x}-\underline{x}\,')$$

$$\int d^3p\,|\,\underline{p}\,><\underline{p}\,|=1$$

$$\int d^3x\,|\,\underline{x}\,><\underline{x}\,|=1$$

$$<\underline{x}\,|\,\underline{p}\,>=\left(\frac{1}{2\pi\hbar}\right)^{3/2}e^{i\underline{x}.\underline{p}/\hbar}$$

The Lipmann Schwinger Equation – operator formalism

$$H_0 \, | \phi >= E \, | \phi > \quad \text{stationary state}$$

$$H_0 + V \, | \psi >= E \, | \psi > \quad (2.1)$$

Solution to (2.1)

$$| \psi >=| \phi > + \frac{1}{E - H_0} V \, | \psi > \qquad (2.2)$$

(up to complications coming from singular nature of the inverse operator $\frac{1}{E - H_0}$)

We will prove a consistent regularisation procedure is

$$| \psi_{\pm} >=| \phi > + \frac{1}{E - H_0 \pm i\varepsilon} V \, | \psi_{\pm} > \qquad (2.3)$$

$$|\psi_\pm> = |\phi> + \frac{1}{E-H_0 \pm i\varepsilon} V |\psi_\pm> \qquad (2.3)$$

In position basis

$$<\underline{x}|\psi_\pm> = <\underline{x}|\phi> + <\underline{x}|\frac{1}{E-H_0 \pm i\varepsilon} V |\psi_\pm>$$

$$\psi_\pm(\underline{x}) = \phi(\underline{x}) + \int d^3x' <\underline{x}|\frac{1}{E-H_0 \pm i\varepsilon}|\underline{x}'><\underline{x}'|V|\psi_\pm>$$

$$<\underline{x}'|V|\psi_\pm> = \int d^3x'' <\underline{x}'|V|\underline{x}''><\underline{x}''|\psi_\pm>$$

$$= \int d^3x'' V(\underline{x}')\delta^3(\underline{x}'-\underline{x}'')\psi_\pm(\underline{x}'')$$

$$= V(\underline{x}')\psi_\pm(\underline{x}')$$

$<\underline{x}'|V|\underline{x}''> = V(\underline{x}')\delta^3(\underline{x}'-\underline{x}'')$ if V local

$$\psi_\pm(\underline{x}) = \phi(\underline{x}) + \int d^3x' <\underline{x}|\frac{1}{E-H_0 \pm i\varepsilon}|\underline{x}'> V(x')\psi_\pm(x)$$

The Green function

$$\psi_\pm(\underline{x}) \; = \phi(\underline{x}) + \int d^3x' < \underline{x} \,|\, \frac{1}{E - H_0 \pm i\varepsilon} \,|\, \underline{x}' > V(x')\psi_\pm(\underline{x}')$$

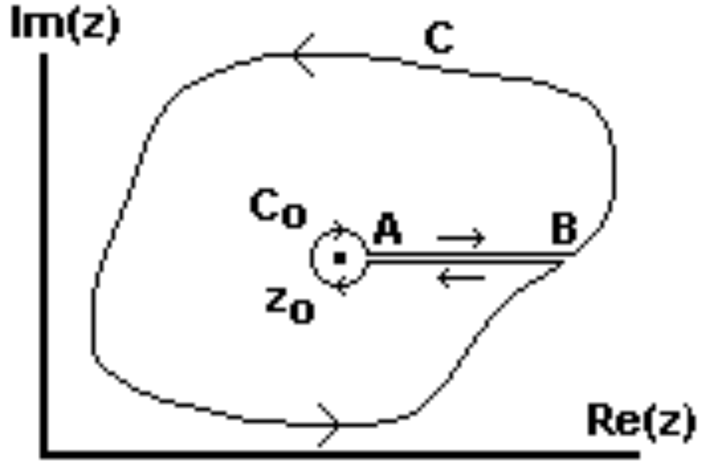
$$c.f. \quad \mathrm{v}_{\mathrm{k}}^{\mathrm{diffractive}}(\underline{r}) = e^{ikz} + \int d^3r' \frac{2\mu}{\hbar^2} G_+(\underline{r} - \underline{r}')\, V(\underline{r}')\, \mathrm{v}_{\mathrm{k}}^{\mathrm{diffractive}}(\underline{r}') \qquad (1.14)$$

$$\Rightarrow \; G_\pm(\underline{x}, \underline{x}') = \frac{\hbar^2}{2\mu} < \underline{x} \,|\, \frac{1}{E - H_0 \pm i\varepsilon} \,|\, \underline{x}' > \qquad (2.5)$$

$$? = -\frac{1}{4\pi} \frac{e^{\pm ik|\underline{x} - \underline{x}'|}}{|\underline{x} - \underline{x}'|} \qquad (1.15) \qquad \left(E \equiv \frac{\hbar^2 k^2}{2\mu} \right)$$

Cauchy's integral formula

$$f(a) = \oint_C \frac{f(z)}{(z-a)} dz$$



Residue theorem

See e.g. Boas P682

$$\oint_C f(z)\,dz = 2\pi i \sum_i \text{Residue}_i$$