"Stationary states" – definite energy E

$i\hbar \frac{\partial \psi}{\partial t}$

$$-\frac{\hbar^2}{2\mu}\nabla_x^{\,2}\psi + \mathrm{V}(x,y,z)\psi = E\psi \equiv \frac{\hbar^2 k^2}{2\mu}\psi \qquad (1.2)$$

$V(\underline{r}) = \frac{\hbar^2}{2\mu} U(\underline{r})$

$$\left[\nabla^2 + k^2 - U(\underline{r})\right]\psi(\underline{r}) = 0 \qquad (1.3)$$

Asymptotic form (V=0 region) of stationary scattering states : scattering amplitude

$$\mathrm{v}_{\mathrm{k}}^{\mathrm{diffractive}}(\underline{r}) \underset{r\to\infty}{\sim} Ae^{ikz} + f_k(\theta,\phi)\,\frac{e^{ikr}}{r} \qquad (1.5)$$

Need to calculate scattering amplitude

Calculation of the scattering amplitude

Integral scattering equation (Green functions)

$$\left(\nabla^2 + k^2\right)\psi_k(\underline{r}) = U(\underline{r})\psi_k(\underline{r}) \qquad (1.3)$$

Green function

$$\left(\nabla^2 + k^2\right)G(\underline{r}) = \delta(\underline{r}) \qquad (1.10)$$

$$\psi_k(\underline{r}) = \psi_{k0}(\underline{r}) + \int d^3 r' \, G(\underline{r} - \underline{r}')\, U(\underline{r}')\, \psi_k(\underline{r}') \qquad (1.11)$$

Solution to (1.3)

$$\left(\nabla^2 + k^2\right)\psi_{k0}(\underline{r}) = 0 \qquad (1.12)$$

The Green function has the form:

$$G_{\pm}(\underline{r}) = -\frac{1}{4\pi}\frac{e^{\pm ikr}}{r} \qquad (1.13)$$

$$\nabla^2\left(\frac{1}{r}\right) = -4\pi\delta^3(\mathbf{r})$$

$$\nabla^2\left(e^{\pm ikr}\right) = -k^2 e^{\pm ikr} \pm \frac{2ik}{r} e^{\pm ikr}$$

$$\nabla^2\left(\frac{e^{\pm ikr}}{r}\right) = \frac{1}{r}\nabla^2 e^{\pm ikr} + e^{\pm ikr}\nabla^2\left(\frac{1}{r}\right) + 2\nabla\left(\frac{1}{r}\right).\nabla e^{\pm ikr}$$

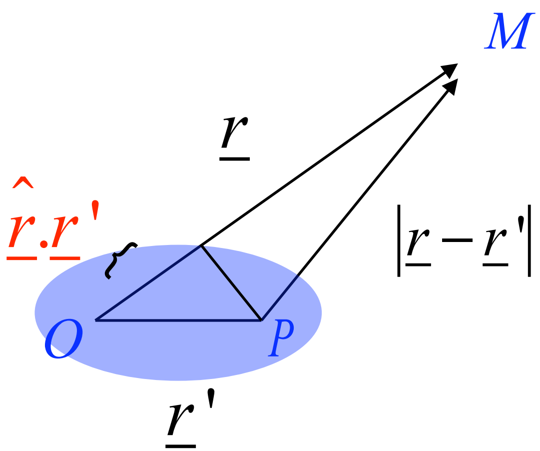
$$= \left(-\frac{k^2}{r} \pm \frac{2ik}{r^2} - 4\pi\delta^3(\mathbf{r}) - \frac{2}{r^2}(\pm ik) + \frac{k^2}{r}\right) e^{\pm ikr}$$

$$\left(\nabla^2 + k^2\right)\frac{e^{\pm ikr}}{r} = -4\pi e^{\pm ikr}\delta^3(\mathbf{r}) = -4\pi\delta^3(\mathbf{r})$$

$$G_{\pm}(\underline{r}) = -\frac{1}{4\pi}\frac{e^{\pm ikr}}{r}$$

$$\mathrm{v}_{\mathrm{k}}^{\mathrm{diffractive}}(\underline{r}) = e^{ikz} + \int d^3r'\, G_{+}(\underline{r}-\underline{r}')\, U(\underline{r}')\, \mathrm{v}_{\mathrm{k}}^{\mathrm{diffractive}}(\underline{r}') \qquad (1.14)$$

Will show that this gives
The correct asymptotic
behaviour (boundary condition)



$$|\underline{r}-\underline{r}'| \simeq |\underline{r}| - \hat{\underline{r}}.\underline{r}' \equiv r - \hat{\underline{r}}.\underline{r}'$$

$$G_{\pm}(\underline{r}-\underline{r}') = -\frac{1}{4\pi}\frac{e^{\pm ik|\underline{r}-\underline{r}'|}}{|\underline{r}-\underline{r}'|} \simeq -\frac{1}{4\pi}\frac{e^{\pm ikr}}{r}e^{\mp ik\hat{\underline{r}}.\underline{r}'} \qquad (1.15)$$

$$\mathrm{v}_{\mathrm{k}}^{\text{diffractive}}(\underline{r}) = e^{ikz} + \int d^3r' G_+(\underline{r}-\underline{r}')\, U(\underline{r}')\, \mathrm{v}_{\mathrm{k}}^{\text{diffractive}}(\underline{r}') \qquad (1.14)$$

$$\sim e^{ikz} - \frac{1}{4\pi}\frac{e^{+ikr}}{r}\int d^3r' e^{-ik\hat{\underline{r}}.\underline{r}'}\, U(\underline{r}')\, \mathrm{v}_{\mathrm{k}}^{\text{diffractive}}(\underline{r}')$$

$$\equiv e^{ikz} + f(\theta,\phi)\frac{e^{ikr}}{r} \qquad (1.5)$$

$$G_\pm(\underline{r}-\underline{r}') = -\frac{1}{4\pi}\frac{e^{\pm ikr}}{r} e^{\mp ik\hat{\underline{r}}.\underline{r}'}$$

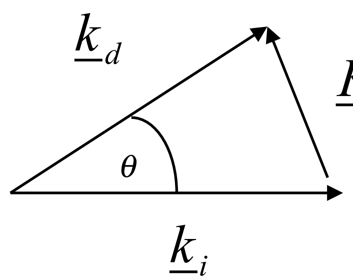
only angular dependence

Stationary scattering state

Scattering amplitude

$$f(\theta,\phi) = -\frac{1}{4\pi}\int d^3r' e^{-ik\hat{\underline{r}}.\underline{r}'}\, U(\underline{r}')\, \mathrm{v}_{\mathrm{k}}^{\text{diffractive}}(\underline{r}') \qquad (1.16)$$

$$\underline{k}_d \equiv k\hat{\underline{r}}$$



$$\underline{K} = \underline{k}_d - \underline{k}_i$$ Scattering or transferred wave vector (1.17)

→ z

$$|\underline{k}_d| = |\underline{k}_i| = k \ \text{(elastic scattering)}$$

The Born Approximation

$$\mathrm{v}_{\mathrm{k}}^{\mathrm{diffractive}}(\underline{r}) = e^{i\underline{k}_i.\underline{r}} + \int d^3r' G_+(\underline{r}-\underline{r}')\, U(\underline{r}')\, \mathrm{v}_{\mathrm{k}}^{\mathrm{diffractive}}(\underline{r}') \qquad (1.14)$$

$$\mathrm{v}_{\mathrm{k}}^{\mathrm{diffractive}}(\underline{r}') = e^{i\underline{k}_i.\underline{r}'} + \int d^3r'' G_+(\underline{r}'-\underline{r}'')\, U(\underline{r}'')\, \mathrm{v}_{\mathrm{k}}^{\mathrm{diffractive}}(\underline{r}'')$$

Born Expansion

Born approximation

$$\mathrm{v}_{\mathrm{k}}^{\mathrm{diffractive}}(\underline{r}) = e^{i\underline{k}_i.\underline{r}} + \underbrace{\int d^3r' G_+(\underline{r}-\underline{r}')\, U(\underline{r}') e^{i\underline{k}_i.\underline{r}'}}$$

$$+ \int d^3r' \int d^3r'' G_+(\underline{r}-\underline{r}')\, U(\underline{r}') G_+(\underline{r}'-\underline{r}'') U(\underline{r}'') \mathrm{v}_{\mathrm{k}}^{\mathrm{diffractive}}(\underline{r}'') \qquad (1.18)$$

Repeat insertion

$$f(\theta,\phi) = -\frac{1}{4\pi}\int d^3r' e^{-ik\hat{\underline{r}}.\underline{r}'}\, U(\underline{r}')\, \mathrm{v}_{\mathrm{k}}^{\mathrm{diffractive}}(\underline{r}')$$

$$f^{Born}(\theta,\phi) = -\frac{1}{4\pi}\int d^3r' e^{-ik\hat{\underline{r}}.\underline{r}'}\, U(\underline{r}')\, e^{i\underline{k}_i.\underline{r}'} \qquad (1.19)$$