Dirac's derivation

Look for equation *linear* in energy

$$H\psi = (\underline{\alpha}.\underline{P} + \beta m)\psi$$

$$H^2\psi = (\alpha_i P_i + \beta m)(\alpha_j P_j + \beta m)\psi = (\underline{P}^2 + m^2)\psi$$

Summation convention

$$(\alpha_i\alpha_j + \alpha_j\alpha_i) = 2\delta_{ij}$$

$$(\alpha_i\beta + \beta\alpha_i) = 0, \quad \beta^2 = 1$$

$$\underline{\alpha} = \begin{pmatrix} 0 & \underline{\sigma} \\ \underline{\sigma} & 0 \end{pmatrix} \quad \beta = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}$$

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

In operator notation

$$i\frac{\partial}{\partial t}\psi = (-i\underline{\alpha}.\underline{\nabla} + \beta m)\psi$$

Premultiply both sides by β

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In operator notation

$$i\beta\frac{\partial}{\partial t}\psi = (-i\beta\underline{\alpha}.\underline{\nabla} + \beta\beta m)\psi$$

$$(i\gamma_\mu\partial^\mu - m)\psi = 0$$

$$\gamma^\mu = (\beta, \beta\underline{\alpha})$$

Dirac-Pauli basis

$$\gamma^\mu = (\beta, \beta\underline{\alpha})$$

$$\underline{\alpha} = \begin{pmatrix} 0 & \underline{\sigma} \\ \underline{\sigma} & 0 \end{pmatrix} \quad \beta = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}$$

$$\gamma_0 = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix} \quad \gamma_i = \begin{pmatrix} 0 & \underline{\sigma} \\ -\underline{\sigma} & 0 \end{pmatrix} \qquad \gamma_5 = i\gamma^0\gamma^1\gamma^2\gamma^3 = \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix}$$

Weyl basis

$$\underline{\alpha} = \begin{pmatrix} -\underline{\sigma} & 0 \\ 0 & \underline{\sigma} \end{pmatrix} \quad \beta = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}$$

$$\gamma_0 = \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix}, \quad \gamma_i = \begin{bmatrix} 0 & \sigma_i \\ -\sigma_i & o \end{bmatrix}, \quad \gamma_5 = i\gamma^0\gamma^1\gamma^2\gamma^3 = \begin{bmatrix} -I & 0 \\ 0 & I \end{bmatrix}$$

$$\{\gamma^\mu, \gamma^\nu\} = \gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2g^{\mu\nu}$$

Free particle solution

$$\left(i\gamma_\mu \partial^\mu - m\right)\psi = 0 \qquad \psi = e^{-ip.x} u(p)$$

$$\left(\gamma_\mu p^\mu - m\right)u(p) \equiv \left(\not{p} - m\right)u(p) = 0$$

More convenient to use

$$Hu = \left(\underline{\alpha}.\underline{P} + \beta m\right)u = Eu \qquad \underline{\alpha} = \begin{pmatrix} 0 & \underline{\sigma} \\ \underline{\sigma} & 0 \end{pmatrix} \quad \beta = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}$$

$$\begin{pmatrix} mI & \underline{\sigma}.\underline{p} \\ \underline{\sigma}.\underline{p} & -mI \end{pmatrix}\begin{pmatrix} u_A \\ u_B \end{pmatrix} = E\begin{pmatrix} u_A \\ u_B \end{pmatrix}$$

$$\underline{\sigma}.\underline{p}\, u_B = (E - m)u_A$$

$$\underline{\sigma}.\underline{p}\, u_A = (E + m)u_B$$

$$\underline{\sigma}.\underline{p}\, u_B = (E-m)u_A$$
$$\underline{\sigma}.\underline{p}\, u_A = (E+m)u_B$$

$If\ \underline{p}=0,\ E=+m$

$(E+m)u_B = 0 \Rightarrow u_B = 0$

For the 2 E>0 solutions , we may take $u_A{}^{(s)} = \chi^{(s)}, \quad \chi^{(1)} = \begin{pmatrix}1\\0\end{pmatrix} \quad \chi^{(2)} = \begin{pmatrix}0\\1\end{pmatrix}$

$$u_B{}^{(s)} = \frac{\underline{\sigma}.\underline{p}}{E+m}\chi^{(s)}$$

Positive energy 4-spinor solutions of Dirac's equation

$$u^{(s)} = N\begin{pmatrix}\chi^{(s)}\\ \dfrac{\underline{\sigma}.\underline{p}}{E+m}\chi^{(s)}\end{pmatrix}, \quad E>0, \quad s=1,2$$

For the 2 E<0 solutions , we may take $u_B^{(s)} = \chi^{(s)}$,

$$u_A^{(s)} = \frac{\underline{\sigma}.\underline{p}}{E-m}\chi^{(s)} = -\frac{\underline{\sigma}.\underline{p}}{|E|+m}\chi^{(s)}$$

$$\underline{\sigma}.\underline{p}\, u_B = (E-m)u_A$$
$$\underline{\sigma}.\underline{p}\, u_A = (E+m)u_B$$

Negative energy 4-spinor solutions of Dirac's equation

$$u^{(s+2)} = N'\begin{pmatrix} -\frac{\underline{\sigma}.\underline{p}}{E+m}\chi^{(s)} \\ \chi^{(s)} \end{pmatrix}, \quad E<0, \quad s=1,2$$

Orthonormal states

$$u^{(r)\dagger}u^{(s)} = 0, \quad r \neq s$$

Non-relativistic correspondance

$$\psi^{(1)} = e^{-(imc^2/\hbar)t}\begin{pmatrix}1\\0\\0\\0\end{pmatrix}, \quad \psi^{(2)} = e^{-(imc^2/\hbar)t}\begin{pmatrix}0\\1\\0\\0\end{pmatrix}, \quad \psi^{(3)} = e^{+(imc^2/\hbar)t}\begin{pmatrix}0\\0\\1\\0\end{pmatrix}, \quad \psi^{(4)} = e^{+(imc^2/\hbar)t}\begin{pmatrix}0\\0\\0\\1\end{pmatrix}$$

Spin – for state at rest $\underline{p} = 0$

$$\psi^{(1)} = e^{-(imc^2/\hbar)t}\begin{pmatrix}1\\0\\0\\0\end{pmatrix},\quad \psi^{(2)} = e^{-(imc^2/\hbar)t}\begin{pmatrix}0\\1\\0\\0\end{pmatrix},\quad \psi^{(3)} = e^{+(imc^2/\hbar)t}\begin{pmatrix}0\\0\\1\\0\end{pmatrix},\quad \psi^{(4)} = e^{+(imc^2/\hbar)t}\begin{pmatrix}0\\0\\0\\1\end{pmatrix}$$

Since we have a (two-fold) degeneracy there must be some operator which commutes with the energy operator and whose eigenvalues label the two states

$$\Sigma^3 \equiv \begin{pmatrix}\sigma^3 & 0\\ 0 & \sigma^3\end{pmatrix} = \begin{pmatrix}1 & 0 & 0 & 0\\ 0 & -1 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & -1\end{pmatrix}$$

Eigenvalues ± 1 $\qquad \Sigma^3\psi^{(1,2)} = \pm\psi^{(1,2)}$

$$\underline{\Sigma} \equiv \begin{pmatrix}\underline{\sigma} & 0\\ 0 & \underline{\sigma}\end{pmatrix}$$

$\left(\frac{1}{2}\hbar\underline{\Sigma}\right)^2 = \frac{3}{4}\hbar^2 I,\quad \frac{1}{2}\hbar\underline{\Sigma}$ has eigenvalues $\pm\frac{1}{2}\hbar$

$\Rightarrow$ **$\frac{1}{2}\hbar\underline{\Sigma}$ is spin operator $\underline{S}$ corresponding to $S = \frac{1}{2}$**

Spin – for state NOT at rest $\underline{p} \neq 0$

$\tfrac{1}{2}\hbar\underline{\Sigma}$ no longer a commuting observable $\left[\tfrac{1}{2}\hbar\underline{\Sigma}, H\right] = \left[\tfrac{1}{2}\hbar\underline{\Sigma}, \underline{\alpha}.\underline{P} + \beta m\right] \neq 0$

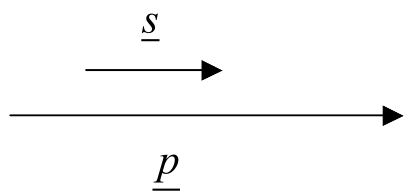
$$\underline{\alpha} = \begin{pmatrix} 0 & \underline{\sigma} \\ \underline{\sigma} & 0 \end{pmatrix} \quad \beta = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}$$

Helicity

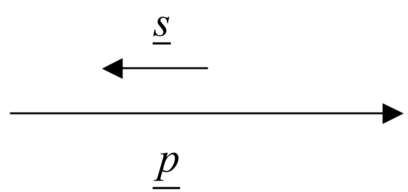
$$\underline{\Sigma}.\widehat{\underline{p}} \equiv \frac{1}{2}\hbar \begin{pmatrix} \underline{\sigma}.\widehat{\underline{p}} & 0 \\ 0 & \underline{\sigma}.\widehat{\underline{p}} \end{pmatrix}, \qquad \widehat{\underline{p}} = \frac{\underline{p}}{|\underline{p}|}$$

Eigenvalues

$+\frac{1}{2}\hbar$



$-\frac{1}{2}\hbar$



(More generally, in arbitrary frame, spin given by boosting result at rest - $s^{\mu} = (0, \underline{s}) \quad \Rightarrow \quad s'^{\mu} = \Lambda^{\mu}_{\nu} s^{\nu}$)

Inclusion of electromagnetic interaction

$$p^\mu \rightarrow p^\mu - \frac{e}{c}A^\mu \qquad A^\mu = (V, \underline{A})$$

$$\Rightarrow \quad i\frac{\partial}{\partial t}\psi = \left(c\underline{\alpha}.\left(\underline{p} - \frac{e}{c}\underline{A}\right) + \beta mc^2 + eV \right)\psi$$

$$\underline{\alpha} = \begin{pmatrix} 0 & \underline{\sigma} \\ \underline{\sigma} & 0 \end{pmatrix} \quad \beta = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}$$

$$H = H_0 + H'; \qquad H' = -e\underline{\alpha}.\underline{A} + eV$$

$$\psi = \begin{pmatrix} \tilde{\phi} \\ \tilde{\psi} \end{pmatrix}$$

$$i\hbar\frac{\partial}{\partial t}\begin{pmatrix} \tilde{\phi} \\ \tilde{\psi} \end{pmatrix} = \left(c\underline{\alpha}.\left(\underline{p} - \frac{e}{c}\underline{A}\right) + \beta mc^2 + eV \right)\begin{pmatrix} \tilde{\phi} \\ \tilde{\psi} \end{pmatrix}$$

$$i\hbar\frac{\partial}{\partial t}\begin{pmatrix} \tilde{\phi} \\ \tilde{\psi} \end{pmatrix} = c\underline{\sigma}.\underline{\pi}\begin{pmatrix} \tilde{\psi} \\ \tilde{\phi} \end{pmatrix} + mc^2\begin{pmatrix} \tilde{\phi} \\ -\tilde{\psi} \end{pmatrix} + eV\begin{pmatrix} \tilde{\phi} \\ \tilde{\psi} \end{pmatrix}$$

$$\underline{\pi} = \underline{p} - \frac{e}{c}\underline{A} = -i\hbar\underline{\nabla} - \frac{e}{c}\underline{A}$$

$$i\hbar\frac{\partial}{\partial t}\begin{pmatrix}\tilde{\phi}\\ \tilde{\psi}\end{pmatrix} = c\underline{\sigma}.\underline{\pi}\begin{pmatrix}\tilde{\psi}\\ \tilde{\phi}\end{pmatrix} + mc^2\begin{pmatrix}\tilde{\phi}\\ -\tilde{\psi}\end{pmatrix} + eV\begin{pmatrix}\tilde{\phi}\\ \tilde{\psi}\end{pmatrix}$$

$$\underline{\pi} = \underline{p} - \frac{e}{c}\underline{A} = -i\hbar\underline{\nabla} - \frac{e}{c}\underline{A}$$

Nonrelativistic limit

Dominant time dependence

$$\begin{pmatrix}\tilde{\phi}\\ \tilde{\psi}\end{pmatrix} = e^{-\left(imc^2/\hbar\right)t}\begin{pmatrix}\phi\\ \psi\end{pmatrix}$$

$$u^{(s)} = N\begin{pmatrix}\chi^{(s)}\\ \frac{\underline{\sigma}.\underline{p}}{E+m}\chi^{(s)}\end{pmatrix}, \quad E>0$$

large component

small component

$$i\hbar\frac{\partial}{\partial t}\begin{pmatrix}\phi\\ \psi\end{pmatrix} \simeq c\underline{\sigma}.\underline{\pi}\begin{pmatrix}\psi\\ \phi\end{pmatrix} - 2mc^2\begin{pmatrix}0\\ \psi\end{pmatrix} + eV\begin{pmatrix}\phi\\ \psi\end{pmatrix}$$

$$i\hbar\frac{\partial\psi}{\partial t} \simeq c\underline{\sigma}.\underline{\pi}\phi - 2mc^2\psi + eV\psi \quad \Rightarrow \quad 0 \simeq c\underline{\sigma}.\underline{\pi}\phi - 2mc^2\psi$$

$$\psi = \frac{\underline{\sigma}.\underline{\pi}}{2mc}\phi$$

$$i\hbar\frac{\partial}{\partial t}\phi \simeq \left(\frac{\underline{\sigma}.\underline{\pi}\,\underline{\sigma}.\underline{\pi}}{2m} + eV\right)\phi$$

$$i\hbar\frac{\partial}{\partial t}\phi \simeq \left(\frac{\underline{\sigma}.\underline{\pi}\,\underline{\sigma}.\underline{\pi}}{2m}+eV\right)\phi$$

$$\underline{\pi} = -i\hbar\underline{\nabla} - \frac{e}{c}\underline{A}$$

$$\sigma.a\;\sigma.b = a.b + i\sigma.\,a\times b \qquad \Rightarrow \qquad \sigma.\pi\;\sigma.\pi = \pi^2 + i\sigma.\,\pi\times\pi$$

$$= \underline{\pi}^2 - \frac{e\hbar}{c}\underline{\sigma}.\underline{B}$$

$$i\hbar\frac{\partial}{\partial t}\phi \simeq \left(\frac{\left(\underline{p}-\frac{e}{c}\underline{A}\right)^2}{2m} - \frac{e\hbar}{2mc}\underline{\sigma}.\underline{B} + eV\right)\phi$$

Gyromagnetic ratio =2 predicted

$$i\hbar\frac{\partial}{\partial t}\phi \simeq \left(\frac{\underline{p}^2}{2m} - \frac{e}{2mc}\underline{B}.\left(\underline{L}+2\underline{S}\right) + eV\right)\phi$$

$$\underline{S} = \frac{1}{2}\hbar\underline{\sigma}$$

Zero mass fermions – the two component neutrino

$$H\psi = (\underline{\alpha}.\underline{P} + \beta m)\psi$$

$$\underline{\alpha} = \begin{pmatrix} -\underline{\sigma} & 0 \\ 0 & \underline{\sigma} \end{pmatrix} \quad \beta = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}$$

Weyl basis

For m=0 …no mixing

$$\psi = \begin{pmatrix} \phi \\ \chi \end{pmatrix}$$

$$E\,\phi = -\underline{\sigma}.\underline{p}\,\phi$$

$$E\,\chi = +\underline{\sigma}.\underline{p}\chi$$

Both have +ve and –ve energy solution

Positive energy solution $E = |\underline{p}|$, $\quad \underline{\sigma}.\widehat{\underline{p}}\,\phi = -\phi$ — Negative helicity neutrino - LH

Negative energy solution $E = -|\underline{p}|$, $\quad \underline{\sigma}.\widehat{\underline{p}}\,\phi = \phi$ — Positive helicity antineutrino - RH

$$\tfrac{1}{2}(1-\gamma_5)\begin{pmatrix} \phi \\ \chi \end{pmatrix} = \begin{pmatrix} \phi \\ 0 \end{pmatrix}$$

Projects LH neutrino

$$\gamma_5 = i\gamma^0\gamma^1\gamma^2\gamma^3 = \begin{bmatrix} -I & 0 \\ 0 & I \end{bmatrix}$$

Parity

$$x^{\mu} = (x^0, x^i) \rightarrow (x^0, -x^i)$$

$$J_i \rightarrow J_i \qquad K_i \rightarrow -K_i$$

$$N_i \rightarrow N_i^{\dagger} \qquad N_i^{\dagger} \rightarrow N_i$$

$$N_i = J_i + iK_i$$

$$\psi = \begin{pmatrix} \psi_L \\ \psi_R \end{pmatrix} \rightarrow \begin{pmatrix} \psi_R \\ \psi_L \end{pmatrix} = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix} \psi = \gamma_0 \psi$$

$$(\gamma_{\mu} \left(p^{\mu} - \frac{e}{c} A^{\mu} \right) - m)\psi = 0$$

Parity invariant

(neutrinos violate parity)