Scattering by a central potential : Method of partial waves

$V(\underline{r}) = V(r)$ - Orbital angular momentum $\underline{\mathrm{L}}$ is constant of motion

$\Rightarrow \quad H, \underline{L}^2, L_Z$ simultaneous eigenstates

- $\phi_{k,l,m}(\underline{r})$ "Partial waves" with angular dependence given by $\mathrm{Y}_{\mathrm{l}}^{\mathrm{m}}(\theta,\phi)$

  $V(r)$ influences only the radial dependence

- At large r partial waves close to eigenfunctions of $\mathrm{H}_0, \underline{L}^2, L_Z$

  $\phi^0{}_{k,l,m}(\underline{r})$ "free spherical waves"

Separation of the wave equation

$$-\frac{\hbar^2}{2\mu}\left[\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right)+\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right)+\frac{1}{r^2\sin\theta}\frac{\partial^2}{\partial\phi^2}\right]\psi+V(r)\psi=E\psi$$

$$\psi(r,\theta,\phi)=R(r)Y(\theta,\phi)$$

$$Y(\theta,\phi)=Y_{l,m}(\theta,\phi)=N_{lm}P_l^m(\cos\theta)\Phi_m(\phi)$$

$$R(r)=R_{k,l}(r)=\frac{u_{k,l}(r)}{r}$$

$$-\frac{\hbar^2}{2\mu}\frac{d^2u_{k,l}}{dr^2}+\left[V(r)+\frac{l(l+1)\hbar^2}{2\mu r^2}\right]u_{k,l}=Eu_{k,l}=\frac{\hbar^2k^2}{2\mu}u_{k,l} \qquad (2.17)$$

Free Spherical waves

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$$\psi(r,\theta,\phi) = R^0_{k,l}(r) Y_{l,m}(\theta,\phi) \equiv \frac{u_{k,l}(r)}{r} Y_{l,m}(\theta,\phi)$$

$$-\frac{\hbar^2}{2\mu}\frac{d^2 u_{k,l}(r)}{dr^2} + \left[\frac{l(l+1)\hbar^2}{2\mu r^2}\right] u_{k,l}(r) = \frac{\hbar^2 k^2}{2\mu} u_{k,l}(r)$$

$$R^0_{k,l}(r) = \sqrt{\frac{2k^2}{\pi}}\, j_l(kr)$$

$$j_l(\rho) = (-1)^l \rho^l \left(\frac{1}{\rho}\frac{d}{d\rho}\right)^l \frac{\sin\rho}{\rho}$$

$$\phi^0_{k,l,m}(\vec{r}) \equiv \langle \vec{r} \,|\, k,l,m \rangle = \sqrt{\frac{2k^2}{\pi}}\, j_l(kr) Y_l^m(\theta,\phi) \qquad (2.13)$$

Orthonormality

$$\langle k,l,m \,|\, k',l',m' \rangle = \tfrac{2}{\pi} kk' \int_0^\infty j_l(kr) j_{l'}(k'r) r^2 dr \int d\Omega\, Y^*_{l,m}(\theta,\phi)\, Y_{l',m'}(\theta,\phi)$$

$$= \qquad \delta(k-k') \qquad \delta_{ll'}\delta mm' \qquad (2.14)$$

Completeness

$$\int_0^\infty dk \sum_{l=0}^{\infty} \sum_{m=-l}^{l} |\phi^0_{k,l,m}\rangle\langle\phi^0_{k,l,m}| = 1 \qquad (2.15)$$

Expansion of a plane wave in terms of free spherical waves

$$<\underline{x}\,|\,0,0,k>=\frac{1}{(2\pi)^{3/2}}e^{ikz}$$

$$e^{ikz}=e^{ikr\cos\theta} \quad \Rightarrow \quad L_Z\,|\,0,0,k>=0 \quad \text{since} \quad L_Z=\frac{\hbar}{i}\frac{\partial}{\partial\phi}$$

$$|\,0,0,k>=\int_0^\infty dk'\sum_{l=0}^{\infty}\sum_{m=-l}^{l}|\,k',l,m><\underbrace{k',l,m\,|\,0,0,k>}_{\propto\,\delta(k-k')\,\delta_{mo}}=\sum_{l=0}^{\infty}c_{kl}\,|\,k,l,0>$$

$$e^{ikz}=\sum_{l=0}^{\infty}i^l\sqrt{4\pi(2l+1)}\,j_l(kr)Y_l^0(\theta)$$

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$$=\sum_{l=0}^{\infty}i^{\text{l}}(2l+1)\,j_l(kr)P_l(\cos\theta) \qquad (2.16)$$

Partial waves in the potential V(r)

$$\psi(r,\theta,\phi) = \frac{u_{k,l}(r)}{r} Y_{l,m}(\theta,\phi)$$

$$-\frac{\hbar^2}{2\mu}\frac{d^2 u_{k,l}(r)}{dr^2} + \left[V(r) + \frac{l(l+1)\hbar^2}{2\mu r^2}\right] u_{k,l}(r) = \frac{\hbar^2 k^2}{2\mu} u_{k,l}(r) \qquad (2.17)$$

$$u_{k,l}(0) = 0$$

Just a 1D problem with a particle of mass μ under influence of a potential

$$V_{effective}(r) = V(r) + \frac{l(l+1)\hbar^2}{2\mu r^2}, \qquad \mathrm{r} > 0$$

$$= \infty, \qquad \mathrm{r} < 0$$

The "phase shift"

$$u_{k,l}(r)\ {}_{r\to\infty}\sim A_l e^{ikr} + B_l e^{-ikr}$$

$L \to R$ $\qquad$ $R \to L$

Reflected current = Incident current $\quad \Rightarrow \quad |A_l| = |B_l|$

$$u_{k,l}(r) = |A_l|\left[e^{ikr}e^{i\phi_A} + e^{-ikr}e^{i\phi_B}\right]$$

$$= |A_l| e^{i\phi} \sin(kr - \beta_l) \qquad (2.19)$$

$$R^0_{k,l}(r) = \sqrt{\frac{2k^2}{\pi}} j_l(kr)_{r\to\infty} \sim \frac{1}{kr}\sin\left(kr - \frac{l\pi}{2}\right) \qquad (2.20)$$

$$\beta_l = \frac{l\pi}{2} - \delta$$

"phase shift"

$$\psi^{diffracted}{}_{k,l,0}(r)_{r\to\infty} \sim C\frac{1}{kr}\sin\left(kr - \frac{l\pi}{2} + \delta_l\right) \qquad (2.21)$$

$$\psi^{diffracted}{}_{k,l,0}(r)_{r\to\infty} \sim C\frac{1}{kr}\sin\left(kr-\frac{l\pi}{2}+\delta_l\right) \qquad (2.21)$$

$$= C\frac{1}{2ik}\left(\frac{e^{-i(kr-\frac{l\pi}{2}+\delta_l)}}{r}-\frac{e^{ikr}}{r}e^{i(-\frac{l\pi}{2}+\delta_l)}\right)$$

c.f. $\mathrm{v}_{\mathrm{k}}^{\mathrm{diffractive}}(\underline{r}) \sim e^{ikz} + f_k(\theta,\phi)\,\frac{e^{ikr}}{r}$ (1.5)

$$e^{ikz} = \sum_{l=0}^{\infty} i^{l}(2l+1)j_l(kr)P_l(\cos\theta)_{\mathrm{r}\to\infty} \propto \sum_{\mathrm{l}=0}^{\infty}\frac{1}{kr}\sin\left(kr-\frac{l\pi}{2}\right) \quad (2.16,\ 2.20)$$

$$\psi^{diffracted}{}_{k,l,0}(r)_{r\to\infty} \sim Ce^{-i\delta_l}\left(\frac{e^{-i(kr-\frac{l\pi}{2})}-e^{i(kr-\frac{l\pi}{2})}}{2ikr}-\frac{e^{ikr}}{r}\frac{1}{k}e^{-i\frac{l\pi}{2}}\frac{\left[e^{2i\delta_l}-1\right]}{2i}\right)$$

$$\psi^{diffracted}(r) = \sum_{l=0}^{\infty} i^{l}(2l+1)\psi^{diffracted}{}_{k,l,0}(r)P_l(\cos\theta);$$

$$f_k(\theta) \equiv \sum_{l=0}^{\infty}(2l+1)f_l(k)P_l(\cos\theta) = \sum_{l=0}^{\infty}(2l+1)\frac{e^{i\delta_l}\sin\delta_l}{k}P_l(\cos\theta);$$

$$f_k(\theta) = \sum_{l=0}^{\infty} (2l+1) f_l(k) P_l(\cos\theta) = \sum_{l=0}^{\infty} (2l+1) \frac{e^{i\delta_l} \sin\delta_l}{k} P_l(\cos\theta)$$

Partial wave amplitude

$$f_l(k) = \frac{e^{i\delta_l} \sin\delta_l}{k}$$

$$\sigma(\theta,\phi) = \sigma(\theta) = \left| f_k(\theta) \right|^2 = \frac{1}{k^2} \sum_{l=0}^{\infty} \left| (2l+1) \sin\delta_l P_l(\cos\theta) \right|^2$$

$$\sigma_{Total} = \int \sigma(\theta,\phi) d\Omega = \frac{4\pi}{k^2} \sum_{l=0}^{\infty} (2l+1) \sin^2\delta_l$$

Determination of the phase shifts

For $r < R$, range of potential

$$-\frac{\hbar^2}{2\mu}\frac{d^2 u_{k,l}(r)}{dr^2} + \left[V(r) + \frac{l(l+1)\hbar^2}{2\mu r^2}\right]u_{k,l}(r) = \frac{\hbar^2 k^2}{2\mu}u_{k,l}(r) \qquad (2.17)$$

For $r > R, V = 0$:

$$\frac{u_{k,l}}{r} = j_l(kr), \quad \eta_l(kr) \quad \text{spherical Bessel functions}$$

Diverges at r=0 but OK for r>R

Define Hankel functions :

$$h_l^{(1)} = j_l + i\eta_l \qquad h_l^{(2)} = j_l - i\eta_l$$

$$h_l^{(1)}\,|_{r\to\infty} \to \frac{e^{i(kr - l\pi/2)}}{ikr} \qquad h_l^{(2)}\,|_{r\to\infty} \to -\frac{e^{-i(kr - l\pi/2)}}{ikr} \qquad (2.26)$$

Determination of the phase shifts

$$\psi^{diffracted}(r) = \sum_{l=0}^{\infty} i^{l}(2l+1)\psi^{diffracted}{}_{k,l,0}(r)P_l(\cos\theta);$$

$$\psi^{diffracted}{}_{k,l,0}(r) = c_l^{(1)}h_l^{(1)}(kr) + c_l^{(2)}h_l^{(2)}(kr), \qquad r > R$$

$$\rightarrow c_l^{(1)}\frac{e^{i(kr-l\pi/2)}}{ikr} - c_l^{(2)}\frac{e^{-i(kr-l\pi/2)}}{ikr}$$

$$c.f.\ \psi^{diffracted}{}_{k,l,0}(r)_{r\to\infty} \sim \frac{1}{2}\left(\frac{e^{-i(kr-\frac{l\pi}{2})}}{ikr} - \frac{e^{ikr}}{ikr}e^{i(-\frac{l\pi}{2}+2\delta_l)}\right) \qquad (2.22)$$

$$\Rightarrow \quad c_l^{(1)} = \frac{1}{2}e^{2i\delta_l}, \qquad c_l^{(1)} = \frac{1}{2}$$

$$\frac{u_{k,l}}{r} \propto \cos\delta_l\ j_l(kr) - \sin\delta_l\ \eta_l(kr)$$

S matrix

$$f_k(\theta) = \sum_{l=0}^{\infty} i^l (2l+1) f_l(k) P_l(\cos\theta) = \sum_{l=0}^{\infty} i^l (2l+1) \frac{e^{i\delta_l}\sin\delta_l}{k} P_l(\cos\theta) \qquad (2.23)$$

S-matrix element

Define $$S_l(k) = 1 + 2ikf_l(k) = 1 + 2ik\frac{e^{i\delta_l}\sin\delta_l}{k}$$

$$\boxed{S_l(k) = e^{2i\delta_l}}$$

Unitarity : $$\left|S_l(k)\right| = 1$$ (conservation of probability for each partial wave)

Optical theorem :

$$\sigma_{Total} = \int \sigma(\theta,\phi)\,d\Omega = \frac{4\pi}{k^2}\sum_{l=0}^{\infty}(2l+1)\sin^2\delta_l$$

$$\operatorname{Im} f_l(k) = \operatorname{Im}\frac{e^{i\delta_l}\sin\delta_l}{k} = \frac{\sin^2\delta_l}{k}$$

$$\boxed{\sigma_{Total} = \frac{4\pi}{k}\operatorname{Im} f_k(\theta = 0)}$$