Geosci 342 Problem Set 4

March 13, 2009

Problem 4.1 For a frictional boundary layer with friction forces represented as a drag $(-u/\tau, -v/\tau)$ within a layer of depth δ, find the vector (u, v) as a function of τ and δ as τ is increased from zero to large values.

The steady linearized equations of motion in the presence of drag are

$$-\frac{u}{\tau} + 2\Omega v = 2\Omega v_g \tag{1}$$

$$-\frac{v}{\tau} - 2\Omega u = -2\Omega u_g \tag{2}$$

Using the result from class, the x-component of the velocity is given by

$$u = -\frac{2\Omega}{4\Omega^2 + \tau^{-2}}\left(\frac{v_g}{\tau} - 2\Omega u_g\right) \tag{3}$$

Following a similar derivation, the y-component of the velocity is obtained from equations (1,2)

$$v = \frac{2\Omega}{4\Omega^2 + \tau^{-2}}\left(\frac{u_g}{\tau} + 2\Omega v_g\right) \tag{4}$$

If we align the coordinate system so that (say) the y-axis is pointing in the direction of the geostrophic wind then we can eliminate u_g from (5,6).

$$u = -\frac{2\Omega}{4\Omega^2 + \tau^{-2}}\left(\frac{v_g}{\tau}\right) \tag{5}$$

$$v = \frac{2\Omega}{4\Omega^2 + \tau^{-2}}\left(2\Omega v_g\right) \tag{6}$$

These equations can be rearranged to highlight the role of the nondimensional parameter $2\Omega\tau$, which is essentially an inverse Ekman number. Form terms involving powers of $2\Omega\tau$ (multiply numerator and denominator by τ^2). Further dividing by the geostrophic wind (v_g) tells us exactly how the wind varies with τ (through the nondimensional number $2\Omega\tau$)

$$\frac{u}{v_g} = -\frac{(2\Omega\tau)}{1 + (2\Omega\tau)^2} \tag{7}$$

$$\frac{v}{v_g} = \frac{(2\Omega\tau)^2}{1 + (2\Omega\tau)^2} \tag{8}$$

This solution has the necessary behavior that the x-component of the wind approaches zero and the y-component approaches its geostrophic value for weak damping ($2\Omega\tau >> 1$). For strong damping τ is small ($2\Omega\tau << 1$) and both u and v are zero. For $2\Omega\tau = 1$ the rotation rate and damping times are comparable and both the x-component and y-component of the velocity have magnitude equal to half the geostrophic wind speed (the actual wind vector is rotated 45° in the cyclonic direction from the geostrophic wind vector). See figure (1).

Note that the geostrophic wind is specified at the top of the boundary layer (a number, independent of boundary layer depth) and by parameterizing the viscous forces as a depth-independent drag law the wind is also independent of depth. In the next problem we will see that the dependence of the wind on τ is analogous to its dependence on depth within the boundary layer, as each layer of fluid exerts an additional drag on the layer below.

Problem 4.2 If the boundary layer frictional force is represented in terms of a viscosity, then the equations for the boundary layer flow become

$$\nu \frac{d^2u}{dz^2} = -2\Omega(v - v_g), \nu \frac{d^2v}{dz^2} = 2\Omega(u - u_g) \tag{9}$$

Solve these equations for the behavior of the vector (u, v) as a function of z, subject to the boundary condition that $u = u_g$ and $v = v_g$ at large z and $u = v = 0$ at $z = 0$... This is a classic problem found in most text books. Adding i times the second equation to the first equation gives

$$\nu \frac{d^2(u + iv)}{dz^2} - 2\Omega i(u + iv) = -2\Omega i(u_g + iv_g) \tag{10}$$

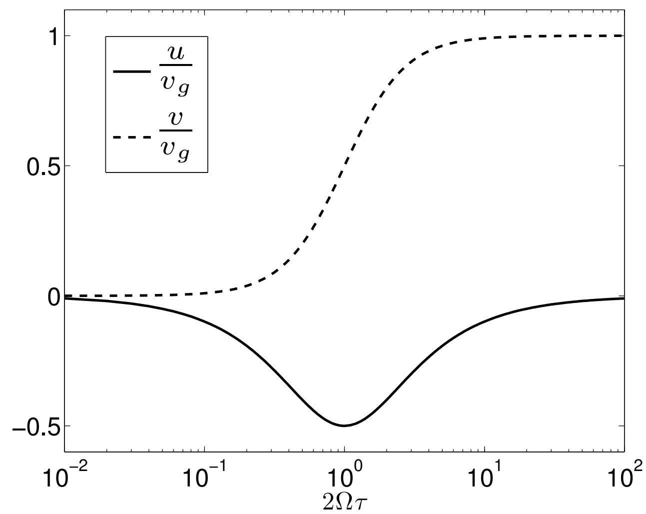

Figure 1: u- and v-components of the wind as a function of the nondimensional damping time $2\Omega\tau$.

Following the usual method (of undetermined coefficients), assume solutions of the form $(u+iv) = e^{\gamma z}$ for the homogeneous problem (zero RHS) to obtain the eigenvalues

$$\gamma = \pm\sqrt{2\Omega i/\nu} \tag{11}$$

Using $\sqrt{i} = \frac{1+i}{\sqrt{2}}$, (11) becomes

$$\gamma = \pm\sqrt{\Omega\nu^{-1}}(1+i) \tag{12}$$

which means the homogeneous solution has exponential and oscillatory behavior. It is easy to find a particular solution if u_g and v_g are independent of depth (which is $u + iv = u_g + iv_g$). Adding this to the above solution gives a general solution

$$\begin{aligned}(u+iv) = Ae^{\sqrt{\Omega\nu^{-1}}z}\Big(\cos\sqrt{\Omega\nu^{-1}}z + i\sin\sqrt{\Omega\nu^{-1}}z\Big) + \\ Be^{-\sqrt{\Omega\nu^{-1}}z}\Big(\cos\sqrt{\Omega\nu^{-1}}z - i\sin\sqrt{\Omega\nu^{-1}}z\Big) + u_g + iv_g\end{aligned} \tag{13}$$

Separating (14) into its real and imaginary parts and setting $A = 0$ by using the boundary condition that the wind approach the geostrophic value as z increases,

$$u = Be^{\sqrt{\Omega\nu^{-1}}z}\cos(\sqrt{\Omega\nu^{-1}}z) + u_g \tag{14}$$
$$v = Be^{-\sqrt{\Omega\nu^{-1}}z}\sin(\sqrt{\Omega\nu^{-1}}z) + v_g \tag{15}$$

The top boundary condition is satisfied for the first equation by setting $B = u_g$. We satisfy the boundary condition in the second equation by rotating the coordinate axes so that the x-axis points in the direction of the geostrophic wind. This is what the mathematics has required by multiplying the v-component velocity by i.

$$u = u_g e^{\sqrt{\Omega\nu^{-1}}z}\cos(\sqrt{\Omega\nu^{-1}}z) + u_g \tag{16}$$
$$v = u_g e^{-\sqrt{\Omega\nu^{-1}}z}\sin(\sqrt{\Omega\nu^{-1}}z) \tag{17}$$

The boundary layer thickeness is determined by the depth at which the wind approaches its geostrophic value. Inspecting equation (17) we see that when $\sin(\sqrt{\Omega\nu^{-1}}z) = 0$ the velocity decays to its geostrophic value (taking $v_g = 0$), which happens when $z = \pi\sqrt{\nu\Omega^{-1}}$.

The divergence is given by taking $\partial u/\partial x, \partial v/\partial y$ of equations (16,17) to get

$$\frac{\partial v}{\partial y} = \zeta_g e^{-\sqrt{\Omega\nu^{-1}}z}\sin(\sqrt{\Omega\nu^{-1}}z) \tag{18}$$

where $\zeta_g = \partial u_g/\partial y$ is the geostrophic relative vorticity in the coordinate system aligned with the geostrophic wind. In differentiating the first equation we use the fact that the geostrophic wind is nondivergent, and $v_g = 0$ in the rotated coordinate system. The vertical velocity is the divergence, vertically integrated to the top of the boundary layer,

$$\begin{aligned} w(z_{top}) = \sqrt{\nu\Omega^{-1}}\frac{1}{2}\Big[&\Big(\sin\sqrt{\Omega\nu^{-1}}z_{top}) - \cos\sqrt{\Omega\nu^{-1}}z_{top}\Big)e^{-\sqrt{\Omega\nu^{-1}}z_{top}} \\ &-\Big(\sin\sqrt{\Omega\nu^{-1}}z(0)) - \cos\sqrt{\Omega\nu^{-1}}z(0)\Big)e^{-\sqrt{\Omega\nu^{-1}}z(0)}\Big] \end{aligned} \tag{19}$$

Plugging in for $z_{top} = \pi\sqrt{\nu\Omega^{-1}}$ and $z(0) = 0$ we get $w(z_{top}) \approx \frac{1}{2}\zeta_g\sqrt{\nu\Omega^{-1}}$. The vertical velocity is proportional to the boundary layer depth, which is in turn dependent on the strength of the viscous dissipation. This result is qualitatively similar to what we obtained in class using the drag law form of viscous dissipation.

Problem 4.3 Including the effects of Ekman damping, the stationary Rossby wave equation for flow over a mountain is

$$U\partial_x \nabla\psi + \beta\partial_x\psi + 2\Omega U\partial_x h/H = -E\nabla\psi \tag{20}$$

Solve this equation for a mountain of shape $h = h_0 \cos kx \sin ly$. The flow is in a channel on the β-plane, and has $v = 0$ at $y = 0$ and $y = \pi/L$...

It is helpful to write the mountain shape as $h = h_0 Re(e^{ikx}) \sin ly$. We try solutions having form similar to the forcing and its derivatives. For $\psi = Ae^{ikx} \sin ly$ we get the following algebraic equation

$$A\Big[Ui(k^3 + kl^2) + \beta ik - E(k^2 + l^2)\Big]e^{ikx} \sin ly = 2\Omega U \frac{h_o}{H} ike^{ikx} \sin ly \tag{21}$$

Then we have an equation for the complex amplitude

$$\tilde{A} = \Big(K^2 - K_s^2 - \frac{iE}{kU}K^2\Big)^{-1} 2\Omega k \frac{h_o}{H} \tag{22}$$

where K_s is the stationary wavenumber and K^2 is the squared magnitude of the total wavenumber vector $(k^2 + l^2)$. Putting the imaginary part in the numerator,

$$\tilde{A} = \frac{\Big(K^2 - K_s^2 - \frac{iE}{kU}K^2\Big) 2\Omega k \frac{h_o}{H}}{(K^2 - K_s^2)^2 + \Big(\frac{E}{kU}K^2\Big)^2} \tag{23}$$

The complex amplitude can be written as $\tilde{A} = A_r + iA_i = |\tilde{A}|e^{i\phi}$ or in other words $\tilde{A} = |\tilde{A}| \cos\phi + i|\tilde{A}| \sin\phi$, where ϕ is whatever it need be to write the complex amplitude in this form and $|\tilde{A}|$ is the square root of the squares of the real and imaginary amplitudes. Using this in our assumed solution, we get $\psi = |\tilde{A}|e^{i\phi} \cos kx \sin ly$. Using $\cos kx = \frac{1}{2}(e^{ikx} - e^{-ikx})$, and multiplying this by $e^{i\phi}$ we get the phase-shifted solution $\psi = |\tilde{A}| \cos(kx + \pi/2) \sin(ly)$, which matches the given boundary conditions.

Collecting real and imaginary parts of the amplitude we get $A_r = |\tilde{A}| \cos\phi$ and $A_i = |\tilde{A}| \sin\phi$, so taking the arctangent of the ratio $A_i/A_r = \tan\phi$ gives the angle ϕ. Plugging the imaginary and real parts of $\tilde{A}$ into this ratio gives

$$\tan\phi = \frac{A_i}{A_r} = \frac{EK^2/kU}{K^2 - K_s^2} \tag{24}$$

When $K = K_s$ we notice that the real part of the amplitude in equation (24) is zero, which means the arctangent of the ratio A_i/A_r will give an angle of 90°. The maximum amplitude still occurs when $K = K_s$, but the damping shifts the wave one-quarter out of phase with the topography and removes the resonant singularity that occurred in the absence of damping. To check the solution, notice that when there is no damping ($E = 0$) the amplitude equation (23) reduces to that of equation (11) of problem set 3, $A = -\frac{2\Omega}{D} h_0/(k_s^2 - k^2)$, except for the sign difference due to the form of the topography.

Problem 4.4 Consider a circular ocean basin on a β-plane. The radius of the ocean basin is r, and it is driven by a wind with profile $U_0 \cos \frac{\pi y}{2r}$, with $y = 0$ taken to be the center of the circle. Assuming Sverdrup balance, find the flow in this ocean basin assuming there is no flow through the boundary on the right-hand side of the basin...

The equation given in class,

$$\beta \frac{\partial \psi}{\partial x} = -\frac{\delta}{H\tau} \frac{\partial u^*}{\partial y} \tag{25}$$

describes a balance between the stretching of vortex columns by Ekman pumping in the ocean surface layer (boundary layer) and the vorticity acquired by meridional fluid motion on a rotating sphere. The Ekman pumping is driven by the wind stress exerted by the atmosphere on the ocean. If we take the wind given in this problem to be the atmospheric surface layer wind (the friction velocity u^*), equation (25) becomes

$$\beta \frac{\partial \psi}{\partial x} = \frac{\delta U_0 \pi}{2H\tau r} \sin \frac{\pi y}{2r} \tag{26}$$

Integrating in x from the semicircle on the right of the domain ($x = \sqrt{r^2 - y^2}$) to an arbitrary x gives

$$\psi(x_{circle}) - \psi(x, y) \propto \sin\left(\frac{y\pi}{2r}\right)\left(\sqrt{r^2 - y^2} - x\right) \tag{27}$$

Taking $\psi = 0$ on the semicircle boundary to satisfy the zero normal flow condition there,

$$\psi(x, y) \propto -\sin\left(\frac{y\pi}{2r}\right)\left(\sqrt{r^2 - y^2} - x\right) \tag{28}$$

Boundary currents must occur on the left side of the domain to satisfy the zero normal flow condition there. See figure 2

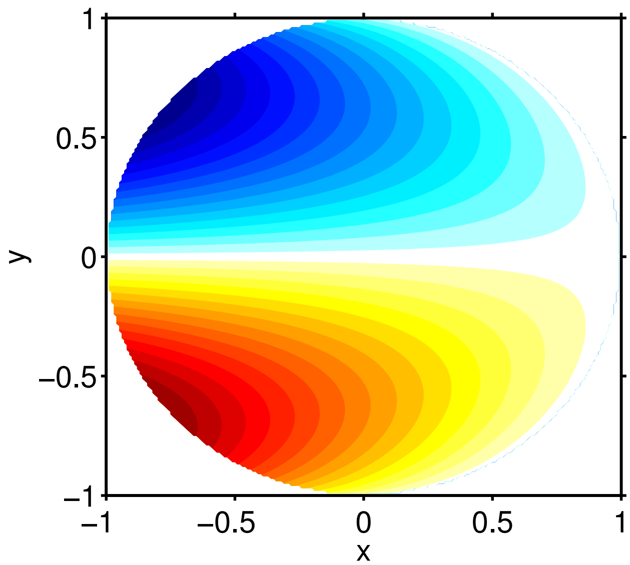

Figure 2: Streamfunction for Sverdrup balance in a circular basin.