Geosci 342 Problem Set 3

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Problem 3.1 Consider a fluid on a rotating plane ($\beta = 0$) which is initially at rest. There is a ridge $h(x)$ at the bottom boundary, with $h = h_0 \exp(-x^2/L^2)$. At time $t = 0$ a uniform flow U in the x-direction is suddenly turned on and the fluid begins to ascend the upwind slope and descend the downwind slope.

(a) After a short time ΔT has passed, what does the vorticity look like over the ridge? Explain your sketch in terms of vortex stretching.

(b) After a long time has passed, what does the vorticity look like over the ridge? Sketch the direction the streamlines bend over the ridge...

From here on all wind velocities and spatial derivatives (including the material derivative) are measured in the rotating reference frame. We'll use the potential vorticity conservation equation derived in class to guide our sketch of the time evolution,

$$\frac{d_g \omega}{dt} = -\frac{2\Omega}{D}\frac{dh(x)}{dt} \tag{1}$$

For this problem the material derivative of the vorticity ($\omega = \omega_{rel} + 2\Omega$) is just the material derivative of the relative vorticity (ω_{rel}), because the vorticity due to the rotating plane (2Ω) is constant in time and constant in space ($\beta = 0$), which means the material derivative does not include any advection of Ω by the wind. In problem 3 we will explore what happens when Ω varies in the y-direction.

Writing (1) as the conservation equation used in the second problem set,

$$\frac{d_g}{dt}\left(\omega_{rel} + \frac{2\Omega}{D}h(x)\right) = 0 \tag{2}$$

Consider two columns in the flow, one initially upstream (windward) of the ridge and another initially at the top the ridge. Both columns have $\omega = 0$ at

$t = 0$ because the flow is initially at rest. Potential vorticity conservation tells us that the initially windward column will acquire negative relative vorticity ($\omega < 0$) as it moves from the windward side to the top of the ridge, but as this same column moves from the top down to the lee side of the ridge the relative vorticity will return to what it was on the windward side ($\omega = 0$) since the height returns to $h = 0$.

By contrast, the relative vorticity of the column initially at the top of the ridge will increase ($\omega > 0$) as it moves down the lee side, since the height decreases, creating an area of cyclonic relative vorticity to the lee of the ridge. We conclude that by starting the flow from rest we end up with different regions of the flow having different values of potential vorticity depending on their initial positions relative to ridge. This result follows directly from the conservation of potential vorticity (equation 2).

To see what happens to the area of cyclonic rotation at a much later time we can expand the material derivative to include the local time rate of change and the advective change,

$$\left(\frac{\partial}{\partial t} + U\frac{\partial}{\partial x}\right)\left(\omega + \frac{2\Omega}{D}h(x)\right) = 0 \tag{3}$$

This equation says the local time evolution of potential vorticity (first LHS derivative) is determined by the advection of potential vorticity at a point in space by the mean wind U. As the column of cyclonic relative vorticity to the lee of the ridge moves further away from the ridge its potential vorticity is conserved (the second quantity in parentheses is constant); furthermore its relative vorticity is conserved because the surface height does not change away from the ridge. This means the cyclonic relative vorticity to the lee of the ridge will move at the wind speed U.

If the advection of cyclonic relative vorticity away from the ridge is not intuitively clear, you can also assume solutions of the form

$$\psi' = e^{i(kx-\omega t)} \tag{4}$$

Plugging equation (4) into equation (3) for the region away from the ridge ($h = 0$) and using $\omega = \partial^2\psi'/\partial x^2$ gives,

$$(-\omega k^2 + Uk^3)e^{i(kx-\omega t)} = 0 \tag{5}$$

which requires

$$c = \frac{\omega}{k} = U \tag{6}$$

This means the area of cyclonic relative vorticity on the lee side of the ridge can be represented as a Fourier series of solutions of the form (4) that travel together at a speed $c = U$, maintaining the shape of the feature as it moves. After this feature moves away from the ridge the relative vorticity over the ridge is then determined entirely by the steady state solution, which is qualitatively similar to that in problem 6 of the second problem set.

Problem 3.2 For midlatitude motions on Earth, the effective value of β is about $10^{-11}\ m^{-1}\ s^{-1}$, in dimensional terms. Consider a wave in a current of $30\ m\ s^{-1}$ which is confined in a channel with width such that the effective north-south wavenumber $l = 10^{-6}\ m^{-1}$. The zonal (east-west) wavelength is $10000\ km$. What is the zonal wavenumber? Using the Rossby wave dispersion relation, determine which direction a ridge or trough of this wave moves, and determine how long it takes for a ridge or trough to move $5000\ km$.

The zonal wavenumber is given by $k = 2\pi/(10000 \cdot 10^3 m) = 6.28 \cdot 10^{-7}\ m^{-1}$. The Rossby wave dispersion relation is

$$c = U - \frac{\beta}{(k^2 + l^2)} \tag{7}$$

Using the values given above, we have a phase speed of

$$c = 30\ m\ s^{-1} - \frac{10^{-11}\ m^{-1}\ s^{-1}}{(6.28 \cdot 10^{-7}\ m^{-1})^2 + (1 \cdot 10^{-6}\ m^{-1})^2} \approx 23\ m\ s^{-1}$$

the phase speed is positive, which means the wave moves toward the east. It takes about 2.5 days ($2.17 \cdot 10^2\ s$) for the wave to move $5000\ km$.

Problem 3.3 Consider easterly flow (i.e. $U < 0$) on the β plane over a sinusoidal mountain $h(x) = h_0 \sin(kx)$. What is the streamfunction field? Is there any phase shift... Now imagine the flow has started up from rest, so that at $t = 0$ the flow is a uniform undisturbed current U. Describe how the superposition of the free and forced wave evolves in time...

The potential vorticity equation (3) with nonzero β is

$$\Big(\frac{\partial}{\partial t} + U\frac{\partial}{\partial x}\Big)\Big(\omega + \frac{2\Omega}{D}h(x)\Big) + \beta\frac{\partial \psi'}{\partial x} = 0$$

Writing this in terms of the perturbation streamfunction and moving the forcing to the RHS,

$$\frac{\partial}{\partial t}\Big(\frac{\partial^2 \psi'}{\partial x^2}\Big) + U\frac{\partial}{\partial x}\Big(\frac{\partial^2 \psi'}{\partial x^2}\Big) + \beta\frac{\partial \psi'}{\partial x} = -U\frac{2\Omega}{D}\frac{\partial h}{\partial x} \tag{8}$$

The forcing, $h(x)$, is constant in time, so the forced solution should satisfy the steady form of equation (8),

$$U\frac{d}{dx}\Big(\frac{d^2\psi'}{dx^2}\Big) + \beta\frac{d\psi'}{dx} = -U\frac{2\Omega}{D}\frac{dh}{dx} \tag{9}$$

Integrating once in x and dividing by U,

$$\frac{d^2\psi'}{dx^2} + k_s^2\psi' = -\frac{2\Omega}{D}h(x) \tag{10}$$

where we use the result from class that the stationary Rossby wavenumber is given by $k_s^2 = \beta/U$. Considering the form of the forcing ($h(x) = h_0\sin(kx)$) we should guess a particular solution $A\sin(kx)$, which when substituted into equation (10) gives

$$A(k_s^2 - k^2) = -\frac{2\Omega}{D}h_0 \tag{11}$$

Notice that $k_s^2 < 0$ because $U < 0$, and $k > 0$ because the topographic wavenumber must be real for a sinusoidal mountain. This means the amplitude of the steady, forced solution is positive ($A > 0$) so that the forced solution is in phase with the topography. Recalling Prof. Nakamura's laboratory demonstration, we know that the beta effect is analogous to a change in the surface height, so it is also analogous to vortex stretching and compression. A steady streamfunction solution in phase with the topography is possible because the cyclonic motion induced by the beta effect as flow moves over a ridge (for example) is balanced by the anticyclonic motion induced by vortex compression and by the advection of relative vorticity over the ridge (see Figure 1).

For the transient free solution we turn to the time dependent equation (8) and set the RHS to zero

$$\frac{\partial}{\partial t}\Big(\frac{\partial^2\psi'}{\partial x^2}\Big) + U\frac{\partial}{\partial x}\Big(\frac{\partial^2\psi'}{\partial x^2}\Big) + \beta\frac{\partial\psi'}{\partial x} = 0 \tag{12}$$

Assuming solutions of the form $\psi' = B\sin(kx - \omega t)$ and substituting into equation (12) gives the equation for the coefficients

$$\Big(-i\omega + ikU\Big)(-k^2) - i\beta k = 0 \tag{13}$$

which, rearranged, gives the dispersion relation

$$c = U - \frac{\beta}{k^2} \tag{14}$$

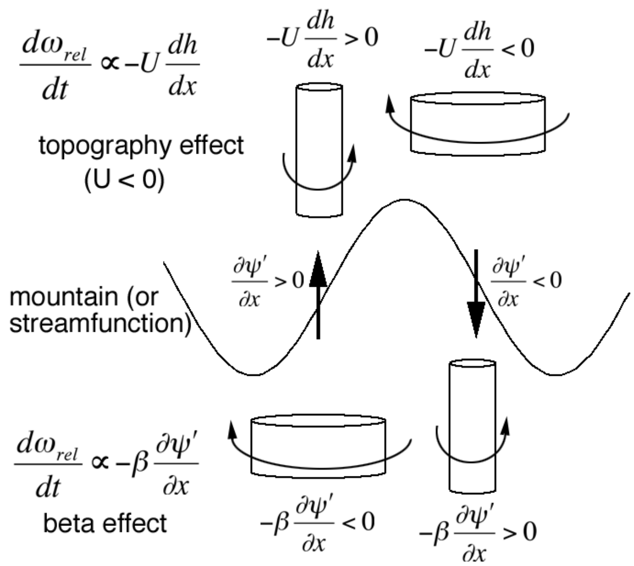

Figure 1: Topography and beta effects in the steady solution for easterly flow over a mountain.

The sine function was chosen to satisfy the condition that the fluid starts at rest. The full solution is the sum of the free and forced parts,

$$\psi' = A\sin(kx) - A\sin(kx - \omega t) \tag{15}$$

where we have taken $B = -A$ to satisfy the condition that the fluid start at rest. From equations (14,15) it is apparent that the free transient solution is initially out of phase with the topography and moves with a negative phase speed ($c < 0$). This rules out the possibility that the transient solution could be resonant (this requires $c = 0, k^2 = k_s^2$). In the long-time limit the solution consists of a transient wave and a steady wave in phase with the topography. In the presence of damping the transient wave would decay over time while the steady forced wave remains. However, since the forced wave cannot be resonant with the topography it would be more difficult to observe in a real system compared to the westerly flow case.